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ASYNCHRONOUS RENDEZVOUS ANALYSIS VIA SET-VALUED CONSENSUS THEORY*

FENG XIAO† AND LONG WANG‡

Abstract. This paper presents the design and analysis result for asynchronous rendezvous control of multiagent systems with continuous-time dynamics and intermittent interactions. The protocol-designing strategies only impose weak restrictions on anticipated-way-point sets (from which the way-points are selected) and path-planning of each agent and can be applied to the networks of arbitrary dimensional subsystems. Explicitly, the anticipated-way-point sets are in a polytope-like form and the path between any two consecutive way-points is required to be included within the minimum convex region covering the two associated anticipated-way-point sets. Under the assumption of directed and switching interaction topology and the assumption of intermittent and asynchronous interactions with time-varying delays, we perform the set-valued consensus analysis on the evolution of anticipated-way-point sets with respect to update times and provide mild sufficient conditions for the solvability of the asynchronous rendezvous problem. The proof techniques rely much on graph theory and nonnegative matrix theory. The obtained result extends greatly the existing work in the literature and several examples demonstrate its broad potential applications. Particularly, additional distributed control rules, different from the circumcenter algorithm, are devised for network connectivity maintenance.

Key words. asynchronous rendezvous, set-valued consensus, intermittent interactions, time-varying delays, connectivity maintenance

AMS subject classifications. 93C65, 93C85, 93C57, 93D99

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1. Introduction.

The objective of this paper is to provide the design and analysis result for asynchronous rendezvous control of networks of dynamic agents with mild constraints, such as switching interaction topology, time-varying information delays, and general agent dynamics. The convergence analysis is within the framework of set-valued consensus theory, developed in this paper.

The rendezvous problem generally falls within the field of decentralized coordination of multiagent systems, which has attracted the attention of a number of researchers and has become a popular research topic because of its theoretical challenges and potential applications, such as cooperative control of unmanned air vehicles, formation control of mobile robots, design of sensor networks, and swarm-based computing. The rendezvous problem was formally studied by Ando et al. in [1], where a distributed memoryless algorithm with the concern of a connectivity-preserving constraint was proposed for a group of mobile robots with limited visibility. The proposed algorithm was later called the *circumcenter algorithm* and its validity for driving all robots to gather at a common location was proved under the assumptions that each robot is able to track its neighbors' positions instantaneously and every pair of robots

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†School of Automation, Beijing Institute of Technology, Beijing 100081, China. Current address: Department of Electrical and Computer Engineering, University of Alberta, Alberta T6G 2V4, Canada (fxiao@ece.ualberta.ca).
‡Intelligent Control Laboratory, Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China (longwang@pku.edu.cn).

is mutually visible. The algorithm was extended to the synchronous and the asynchronous cases with continuous-time dynamics by Lin, Morse, and Anderson in [2] and [3], respectively. The synchronous algorithm was further extended by Conte and Pennesi [4]. There are also some other kinds of cooperative rendezvous algorithms applicable in different cases, such as those for networks of nonholonomic unicycles [5, 6]. To achieve the aim of rendezvous as well as minimize the total travel cost, Litus, Zebrowski, and Vaughan proposed a simple heuristic controller for each agent [7]. Other related work includes rendezvous control via communication over quantized channels [8] or in an organized convergence formation [9].

From a different perspective, rendezvous problem can be viewed as a special case of consensus problem. The latter emphasizes more the abstract agreement quantities, instances of which other than positions include anticipated attitude in multiple spacecraft alignment, velocity in flocking control, and processing rate in distributed task management [10, 11, 12, 13]. The work of this paper is a contribution to the theory of asynchronous consensus and set-valued consensus.

Pioneering work related to asynchronous consensus was conducted by Borkar and Varaiya [14] and Tsitsiklis and Athans [15] in the field of distributed decision-making systems. In [16], Vicsek et al. proposed a simple but interesting discrete-time consensus model of multiple agents moving in the plane with a constant absolute velocity. Each agent's heading is updated using a local rule based on the average of its own heading as well as its neighbors'. In subsequent study, many researchers investigated various extended versions of the Vicsek model and presented their associated convergence analysis [17, 18, 19, 20]. Particularly, Fang and Antsaklis presented asynchronous nonlinear protocols for a generalized Vicsek model and showed that the consensus is reachable under directed and time-varying topologies by the asynchronous iteration methods for nonlinear paracontractions [21]. In their paper, the delayed information of neighbors can be used in the feedback and a robot rendezvous algorithm was presented as an application. In [22], Cao, Morse, and Anderson studied another version of the asynchronous Vicsek model, in which each agent independently updates its desired way-point at the discrete time instants determined by its own clock and then the agent changes its heading from its current value to its desired way-point in a monotonic and at least piecewise-continuous manner. In related literature [23], Xiao and Wang developed the work of [24] and proposed an asynchronous consensus protocol for the systems with intermittent information transmission and time-varying delays. For other work related to time-delayed information, see [25, 26, 27], where the frequency-domain approach and linear matrix inequalities were employed, respectively.

This paper presents a new and general protocol-designing framework for the rendezvous control of networks of multiple agents moving in any dimensional space, whose effectiveness is shown by the newly developed set-valued consensus analysis. Compared with the existing ones, such as the synchronous and asynchronous circumcenter algorithms [2, 3], the decentralized control protocols to be designed here stress their validity in the case with directed switching topology, delayed information, and intermittent interactions, and they are also applicable in the case with asymmetrical information and network connectivity-preserving constraint. In addition, the studied model includes the asynchronous consensus model studied in [23] and also covers the asynchronous Vicsek model [22] and other generalized versions of the Vicsek model, studied, for example, in [17, 18, 28], as its special cases.

In this paper, each agent is associated with a set-valued function in a polytope-like form, called anticipated-way-point set. It is assumed that way-points are selected from anticipated-way-point sets, and the path between any two consecutive way-points is inside the minimum convex region, covering the two associated anticipated-way-point sets. The weak assumptions improve the flexibility of agents' path planning and enlarge the application area of the protocols, such as in the networks of non-holonomic dynamic agents, second-order or higher-order dynamic agents, and heterogeneous agents.

The other contribution of this paper involves the set-valued consensus analysis of anticipated-way-point sets by graph theory and nonnegative matrix theory. We emphasize that the proof techniques are different from the traditional *analytic synchronization* method, employed in [3, 22, 23], although they are all based on some common preliminary works, such as merging the update times of n agents into a single ordered time sequence. One obvious difference between the two approaches is that the evolution process of anticipated-way-point set of each agent is not a traditional dynamical system, whose input does not depend directly on other agents' anticipated-way-point sets. To get the convergence result, we introduce a delay operator and associate the anticipated-way-point set of each agent with n vectors, which enable us to represent the state of the system by nnN vectors in a mathematically equivalent way. The "dimension-expanding" approach converts the set-valued consensus problem into its equivalent discrete-time consensus representation in the traditional sense. The equivalent model follows higher dimensional dynamics and possesses some special network structures, which add much more new content and challenges into consensus analysis. By studying a set of identity-like matrices and introducing a dimension-reducing map, we establish a connection between the interaction topology of the original asynchronous set-valued system and the state matrix of the equivalent augmented system, and by the convergence result on the product of a compact set of infinite stochastic matrices, developed in [20], we provide mild sufficient conditions for the solvability of the asynchronous rendezvous problem.

This paper is organized as follows. The model is set up in section 2. The main result is given in section 3 and its set-valued consensus analysis is postponed to section 4. Simulation examples are given to illustrate the theoretical result in section 5. Finally, concluding remarks are stated in section 6. In the appendix, proofs of some related conclusions are assembled.

2. Problem formulation.

In this section, the model under study is set up. Subsection 2.1 defines detection times and update times and associates them with two assumptions, corresponding to frequency of update actions and maximum allowable time delay of valid information, respectively. Subsection 2.2 gives the definition of interaction topology and discusses its properties. The restrictions on way-points and allowable motion regions between any two consecutive way-points are gathered in subsections 2.3.

2.1. Detection time and update time.

The system studied in this paper consists of n autonomous agents, labeled 1 through n. All these agents share a common state space $\mathbb{R}^N$, $N = 1, 2$, or 3, representing the positions of agents. The state of agent i is denoted by x_i, $i \in \mathcal{I}_n$, where $\mathcal{I}_n$ denotes the index set $\{1, 2, \dots, n\}$. Each agent has a limited sensing range and detects the relative positions of its neighboring agents intermittently. Based on the available detected information, agent i, $i \in \mathcal{I}_n$, calculates and updates its temporary anticipated destination, called the *way-point* [2, 3, 22], at *update times* t_k^i, $k = 0, 1, 2, \dots$. Then it plans its path and moves toward the way-point before its next update time t_{k+1}^i. Suppose that the update time sequences $t_0^i, t_1^i, t_2^i, \dots$, $i \in \mathcal{I}_n$, with $t_0^1 = t_0^2 = \cdots = t_0^n = 0$, satisfy the following assumption:

(A1) There exist positive real numbers $\tau_{\min}$ and $\tau_{\max}$ such that $\tau_{\min} \le t^i_{k+1} - t^i_k \le \tau_{\max}$ for all i, k.

The existence of upper bound $\tau_{\max}$ and lower bound $\tau_{\min}$ in the above assumption guarantees that the frequencies of agent update actions cannot differ much from each other. Similar assumptions can be found in [22, 23]. One counterexample will be given in section 3 to show the necessity of the above constraint in solving the rendezvous problem.

At detection time, for example, t_d, of agent i, agent i may detect the relative positions of all or part of its neighboring agents within its sensing range according to the hardware capability. If agent j is a neighboring agent and the relative position vector $x_j(t_d) - x_i(t_d)$ is obtained by agent i, then at subsequent update time t^i_k, $(x_j(t_d) - x_i(t_d)) - (x_i(t^i_k) - x_i(t_d))$, namely, $x_j(t_d) - x_i(t^i_k)$, becomes the allowable information of agent i, used to calculate its anticipated way-point, where $x_i(t^i_k) - x_i(t_d)$ is the displacement of agent i over the time-interval $[t_d, t^i_k]$. In what follows, data like $x_j(t_d) - x_i(t^i_k)$ will also be called *relative position information*.

Denote the information set of agent i available at update time t^i_k by $\mathcal{A}^i_k$, which includes the data obtained at and before update time t^i_k. Clearly, its current definition does not exclude the case that $\mathcal{A}^i_k$ includes more than one piece of relative position information about the same agent. To decrease the algorithm's complexity and increase the efficiency, the members in $\mathcal{A}^i_k$ should be refined. One possible way to refine is the *most-recent-data* strategy, which requires that if $x_j(t') - x_i(t^i_k)$, $x_j(t'') - x_i(t^i_k) \in \mathcal{A}^i_k$ such that $t' < t''$, then $x_j(t') - x_i(t^i_k)$ should be removed from $\mathcal{A}^i_k$. We emphasize that in $\mathcal{A}^i_k$, the data related to different agents or the data obtained at different detection times are treated as different, although they may have equal numerical values; in other words, the data are distinguished by their symbolic expressions, and the ownerships and detection times are the only basis in data refining. The remainder of this paper supposes that information set $\mathcal{A}^i_k$ has already been refined and dropped the redundant information about the same agent. Furthermore, the information in $\mathcal{A}^i_k$ must be sufficiently new, that is, $\mathcal{A}^i_k$ should satisfy the following assumption:

(A2) If $x_j(t_d) - x_i(t^i_k) \in \mathcal{A}^i_k$, then $0 \le t^i_k - t_d \le \tau_d$, where τ_d is the maximum allowable time-delay.

Note that consecutive information sets $\mathcal{A}^i_k$ and $\mathcal{A}^i_{k+1}$ may intersect with each other.

Finally, we end this subsection with the definition of "asynchronous system." In a distributed system, it is hard to drive all agents to detect information and update their way-points *synchronously* by a common global clock, and thus this paper will mainly study the *asynchronous* case; that is, the detection and update times of each agent are independent of those of others.

2.2. Interaction topology. The underlying interaction topology is mainly determined by agents' positions and sensing ranges. Because of the changing agent positions and headings, the irregularity of sensing ranges, and the existence of obstacles, the unidirectional and time-varying interaction topology is a reasonable assumption. A usual way to represent it is to use a time-dependent directed graph $\mathcal{G}(t)$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ to model the interaction topology, where directed graph $\mathcal{G}$ is defined as an abstract ordered pair, comprising a vertex set $\mathcal{V}(\mathcal{G})$ together with an edge set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$. Agent i is represented by vertex v_i, and the index set of its neighboring agents at time t is denoted by $\mathcal{N}_i(t)$, satisfying the property that $j \in \mathcal{N}_i(t) \iff (v_j, v_i) \in \mathcal{E}(\mathcal{G}(t)) \iff$ agent j is located within the sensing range of agent i. However, by definition, the interaction topology $\mathcal{G}(t)$ represents only the adjacent relation between agents, whereas it cannot indicate whether the neighbors' relative position information is used in the computation of way-points and path planning. To reflect the relation determined by the information usage, we present the following definition.

DEFINITION 2.1 (interaction topology $\mathcal{G}^A(t)$). *The vertices of interaction topology $\mathcal{G}^A(t)$ are $v_1, v_2, \ldots, v_n$, representing agent $1, 2, \ldots, n$, respectively. For any $t \in [t_k^i, t_{k+1}^i)$, $(v_j, v_i) \in \mathcal{E}(\mathcal{G}^A(t))$ if and only if $\mathcal{A}_k^i$ includes the relative position information about agent j, i.e., there exists τ_k^{ij}, $\tau_k^{ij} \geq 0$, such that $x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i) \in \mathcal{A}_k^i$. It is further assumed that each vertex in $\mathcal{G}^A(t)$ has no self-loop, namely, the edge such as (v_i, v_i), $i \in \mathcal{I}_n$. If $(v_j, v_i) \in \mathcal{E}(\mathcal{G}^A(t))$, then agent j is called a neighbor of agent i. The index set of the neighbors of agent i is denoted by $\mathcal{N}_i^A(t)$.*

We will see that the interaction topology $\mathcal{G}^A(t)$ just reflects the effect of obtained information on agents' trajectories. For convenience, denote all elements in $\mathcal{A}_k^i$ by $x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i)$, $j \in \mathcal{N}_i^A(t_k^i)$. Clearly, by assumption (A2),

$$0 \leq \tau_k^{ij} \leq \tau_d.$$

Furthermore, by its definition, if $j \in \mathcal{N}_i^A(t_k^i)$, then $j \in \mathcal{N}_i(t_k^i - \tau_k^{ij})$ and thus

$$\mathcal{N}_i^A(t_k^i) \subset \bigcup_{j \in \mathcal{N}_i^A(t_k^i)} \mathcal{N}_i(t_k^i - \tau_k^{ij}).$$

2.3. Anticipated-way-point sets and allowable motion region. The possible way-point of agent i at update time t_{k+1}^i is chosen from the following *anticipated-way-point set*

$$\mathcal{D}_k^i = \left\{ x_i(t_k^i) + \frac{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \omega_j \left(x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i) \right)}{\sum_{l \in \mathcal{N}_i^A(t_k^i) \cup \{i\}} \omega_l} : \begin{array}{l} x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i) \in \mathcal{A}_k^i, j \in \mathcal{N}_i^A(t_k^i), \\ 0 < W^L \leq \omega_l \leq W^U \text{ for } l \in \mathcal{N}_i^A(t_k^i) \cup \{i\} \end{array} \right\}, \tag{2.1}$$

where parameters ω_l, $l \in \mathcal{N}_i^A(t_k^i) \cup \{i\}$, are called *weighting factors* [18] and their lower bound W^L and upper bound W^U are constant real numbers. Mathematically, the following is required:

(A3) For any $i \in \mathcal{I}_n$ and $k \in \mathbb{N}$, $x_i(t_{k+1}^i) \in \mathcal{D}_k^i$, where the position of $x_i(t_{k+1}^i)$ represents the real way-point.

The property about $\mathcal{D}_k^i$ is characterized by the following lemma.

LEMMA 2.2. *For any $i \in \mathcal{I}_n$ and any $k \in \mathbb{N}$, $\mathcal{D}_k^i$ is a convex compact set and it is also a subset of the convex hull of set $\{x_i(t_k^i), x_j(t_k^i - \tau_k^{ij}), j \in \mathcal{N}_i^A(t_k^i)\}$.*

Proof. See the appendix for the proof. ∎

COROLLARY 2.3. *Let*

$$\tilde{\mathcal{D}}_k^i = \left\{ x_i(t_k^i) + \frac{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \omega_j \left(x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i) \right)}{\sum_{l \in \mathcal{N}_i^A(t_k^i) \cup \{i\}} \omega_l} : \omega_l \text{ equals } W^L \text{ or } W^U \text{ for } l \in \mathcal{N}_i^A(t_k^i) \cup \{i\} \right\}.$$

Then $\mathcal{D}_k^i = co(\tilde{\mathcal{D}}_k^i)$, where $\mathrm{co}(\cdot)$ denotes the convex hull of the considered set.

Proof. See the appendix for the proof. ∎

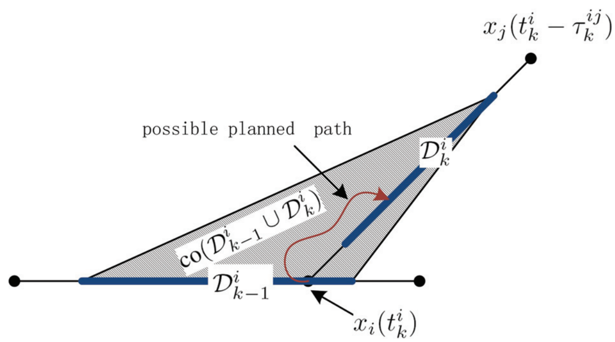

FIG. 2.1. *Anticipated-way-point sets and allowable motion region.*

After update time t_k^i, agent i will move from its current position $x_i(t_k^i)$ toward its next way-point $x(t_{k+1}^i)$, selected from $\mathcal{D}_k^i$. The allowable motion region is described by the following assumption:

(A4) For $t \in [t_k^i, t_{k+1}^i]$

$$x_i(t) \in \text{co}(\mathcal{D}_{k-1}^i \cup \mathcal{D}_k^i).$$

Figure 2.1 shows the anticipated-way-point sets and allowable motion region of agent i when there exists only one neighbor at times t_{k-1}^i and t_k^i.

As we will see, assumption (A4) can be relaxed further. For example, if it is replaced by $x_i(t) \in \text{co}(\mathcal{D}_{k-k^0}^i \cup \mathcal{D}_{k+1-k^0}^i \cup \cdots \cup \mathcal{D}_k^i)$ for $t \in [t_k^i, t_{k+1}^i]$, where k^0 is a given positive integer, then the main result presented in the next section is still obtainable by the similar proof given in section 4.

3. Convergence result. To present the main result, we need several notions in graph theory. A *path* in directed graph $\mathcal{G}$ from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of finite vertices such that $(v_{i_l}, v_{i_{l+1}}) \in \mathcal{E}(\mathcal{G})$ for $l = 1, 2, \ldots, k-1$. Directed graph $\mathcal{G}$ is said to *have a spanning tree* if there exists a vertex, called the *root*, such that it can be connected to any other vertices through paths. If for any i, j, $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ implies that $(v_j, v_i) \in \mathcal{E}(\mathcal{G})$, $\mathcal{G}$ is undirected. And in this case, $\mathcal{G}$ is called to be connected if $\mathcal{G}$ has a spanning tree. The *union* of a group of directed graphs $\mathcal{G}_i$, $i \in \mathcal{I}$, with a common vertex set $\mathcal{V}$ is also a directed graph with the vertex set $\mathcal{V}$ and with the edge set given by $\bigcup_{i \in \mathcal{I}} \mathcal{E}(\mathcal{G}_i)$, where $\mathcal{I}$ is the index set of the group.

With the above preparations, we present the following result.

THEOREM 3.1 (convergence). *Suppose that the system under study satisfies assumptions* (A1) *through* (A4). *If there exists a constant positive real number T such that for all time t^0, the union of the interaction topology $\mathcal{G}^A(t)$ across the time interval $[t^0, t^0 + T]$ always has a spanning tree, then all agents will solve the rendezvous problem asymptotically together, namely, there exists an $x^* \in \mathbb{R}^N$ such that for any $i \in \mathcal{I}_n$, $x_i(t)$ converges to x^* as time evolves.*

In the above theorem, the solvability of the rendezvous problem is indeed the asymptotical consensus of agents' positions $x_i(t)$, $i \in \mathcal{I}_n$. However, note that the state trajectory of variables $x_i(t)$, $i \in \mathcal{I}_n$, between each pair of consecutive update times is hard to determine but is covered by the convex hull of associated anticipated-way-point sets. Thus, it is more natural to describe the rendezvous problem with a *set-valued consensus problem.* Specifically, we will investigate the evolution of anticipated-way-point sets with respect to update times and show that there exists an $x^* \in \mathbb{R}^N$

with the property that for any $\varepsilon > 0$, there exists $k^* \in \mathbb{N}$ such that for any $k \geq k^*$ and any i, we have $\mathcal{D}_k^i \subset \mathcal{B}(x^*, \varepsilon)$, where $\mathcal{B}(x^*, \varepsilon)$ denotes the set of points whose distances to x^* are strictly smaller than ε. Its proof is rather long and relies much on the results from graph theory and nonnegative matrix theory. Therefore, the proof, as an independent part, is presented in the next section.

Relaxable assumptions (A1) through (A4) only describe the restrictions on update times, anticipated-way-point sets and allowable motion region. Thus, the studied system is very general in its form and its instances include a large quantity of multiagent systems with diverse agent dynamics, such as the examples given in the forthcoming subsection. Moreover, the sufficient condition provided in Theorem 3.1 for the solvability of the rendezvous problem is not more conservative than the previous results, such as those provided in [2, 3, 17, 18, 19, 22, 23]. Note that the condition of "periodical union of the interaction graph" can be further relaxed in some special cases; see the work of Moreau on the *bidirectional* interaction case [19]. In addition, the data in $\mathcal{A}_k^i$ are the relative position vectors and independent of the coordinate system of other agents. Thus if the global coordinate is replaced by the local one of each agent, the main result is also establishable.

3.1. Examples in the one-dimensional case. This subsection gives several examples in the one-dimensional case, namely, $N = 1$, to show applications of the main result.

In the first example, the system is assumed to be a synchronous one, namely, $t_k^i = t_k^j$ for all i, j, k, and the detection actions are assumed to coincide with update actions. Then the evolution of agent i's position with respect to update times can be represented by the following discrete-time system:

$$x_i(k+1) = \frac{1}{\sum_{j \in \mathcal{N}_i^A(k) \cup \{i\}} W_{ij}(k)} \sum_{j \in \mathcal{N}_i^A(k) \cup \{i\}} W_{ij}(k)(x_j(k - \tau_k^{ij})), \tag{3.1}$$

where W_{ij}, $j \in \mathcal{N}_i^A(k) \cup \{i\}$, are weighting factors, parameter k in brackets represents time t_k^i, delay terms τ_k^{ij} are nonnegative integers, and $\tau_k^{ii} = 0$. System (3.1) was studied in [28] and covers the simplified Vicsek model studied by Jadbabaie, Lin, and Morse [17] and the extended Vicsek model studied by Ren and Beard [18] as its special cases.

The second example assumes that no delayed information is used and all weighting factors are all equal, namely, $W^L = W^U$. Thus anticipated-way-point set $\mathcal{D}_k^i$ is a singleton $\{x_i(t_{k+1}^i)\}$ and then the studied model becomes the asynchronous Vicsek Model [22], described by

$$\begin{cases} x_i(t_{k+1}^i) = \dfrac{1}{n_i(t_k^i) + 1} \left(\displaystyle\sum_{j \in \mathcal{N}_i^A(t_k^i) \cup \{i\}} x_j(t_k^i) \right), \\ \quad x_i(t) = x_i(t_k^i) + u_i^k(t)(x_i(t_{k+1}^i) - x_i(t_k^i)), \ t \in [t_k^i, t_{k+1}^i), \end{cases}$$

where $n_i(t_k^i)$, $i \in \mathcal{I}_n$, denotes the number of elements contained in $\mathcal{N}_i^A(t_k^i)$ and $u_i^k(t) : [t_k^i, t_{k+1}^i] \to [0, 1]$ is a monotonic and continuous function such that $u_i^k(t_k^i) = 0$ and $u_i^k(t_{k+1}^i) = 1$.

The next example is the asynchronous system given in [23] and governed by

$$\dot{x}_i(t) = \frac{1}{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \alpha_{ij}(t_k^i)} \sum_{j \in \mathcal{N}_i^A(t_k^i)} \alpha_{ij}(t_k^i)\big(x_j(t_k^i - \tau_k^{ij}) - x_i(t)\big), \ t \in [t_k^i, t_{k+1}^i), \quad i \in \mathcal{I}_n. \tag{3.2}$$

In [23], it was assumed that the update time sequences $t_k^i, k \in \mathbb{N}, i \in \mathcal{I}_n$, satisfy assumption (A1), delay terms τ_k^{ij} satisfy assumption (A2), and $\check{\alpha} \le \alpha_{ij}(t_k^i) \le \hat{\alpha}$ for some positive real numbers $\check{\alpha}$ and $\hat{\alpha}$. Next, we show that there exist some positive real numbers W^L and W^U such that assumptions (A3) and (A4) are also satisfied.

Solving (3.2) gives that (see equation (4) in [23])

$$x_i(t) = e^{-(t-t_k^i)} x_i(t_k^i) + (1 - e^{-(t-t_k^i)}) \left(\frac{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \alpha_{ij}(t_k^i) x_j(t_k^i - \tau_k^{ij})}{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \alpha_{ij}(t_k^i)} \right), \quad t \in [t_k^i, t_{k+1}^i]. \tag{3.3}$$

By assumption (A1), $e^{-\tau_{\max}} \le e^{-(t-t_k^i)} \le e^{-\tau_{\min}}$. Define β_j, $j \in \mathcal{I}_n$, by

$$\beta_j = \begin{cases} (1 - e^{t_k^i - t_{k+1}^i}) \alpha_{ij}(t_k^i), & j \in \mathcal{N}_i^A(t_k^i), \\ e^{t_k^i - t_{k+1}^i} \sum_{l \in \mathcal{N}_i^A(t_k^i)} \alpha_{il}(t_k^i), & j = i. \end{cases}$$

Then

$$x_i(t_{k+1}^i) = x_i(t_k^i) + \frac{\sum_{j \in \mathcal{N}_i^A(t_k^i)} \beta_j (x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i))}{\sum_{l \in \mathcal{N}_i^A(t_k^i) \cup \{i\}} \beta_l}$$

and

$$\min\{(1 - e^{-\tau_{\min}})\check{\alpha}, e^{-\tau_{\max}}\check{\alpha}\} \le \beta_j \le \max\{(1 - e^{-\tau_{\max}})\hat{\alpha}, (n-1)e^{-\tau_{\min}}\hat{\alpha}\}, \quad j \in \mathcal{I}_n.$$

Denote $W^L = \min\{(1 - e^{-\tau_{\min}})\check{\alpha}, e^{-\tau_{\max}}\check{\alpha}\}$ and $W^U = \max\{(1 - e^{-\tau_{\max}})\hat{\alpha}, (n-1)e^{-\tau_{\min}}\hat{\alpha}\}$. Then system (3.2) satisfies assumption (A3). By the monotonic property of function $1 - e^{-(t-t_k^i)}$, assumption (A4) is automatically satisfied.

Counterexample without assumption (A1). As shown in the above discussion, Example 1 given in [23] satisfies assumptions (A2) through (A4) other than assumption (A1). The vibration phenomenon of agents' states shows the necessity of the existence of upper bound $\tau_{\max}$ in assumption (A1). From a practical viewpoint, the lower bound $\tau_{\min}$ is also necessary since any agent cannot move from one way-point to the next infinitely rapidly due to the restriction of input saturation. In theory, if the lower bound $\tau_{\min}$ does not exist, then the frequencies of agents' update actions can differ a lot from each other. To construct the counterexample, suppose that $n = 3$, the system satisfies assumptions (A2) through (A4), no delayed information is used, the interaction topology $\mathcal{G}^A(t)$ is switching periodically, and the switching sequence is given as

$$\mathcal{G}^A(t) = \begin{cases} \mathcal{G}_a, & t \in [4k, 4k+1), \\ \mathcal{G}_b, & t \in [4k+1, 4k+2), \\ \mathcal{G}_c, & t \in [4k+2, 4k+3), \\ \mathcal{G}_d, & t \in [4k+3, 4(k+1)), \end{cases} \quad k = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} \mathcal{E}(\mathcal{G}_a) &= \{(v_1, v_2)\}, \\ \mathcal{E}(\mathcal{G}_b) &= \{(v_1, v_2), (v_2, v_1)\}, \\ \mathcal{E}(\mathcal{G}_c) &= \{(v_3, v_2)\}, \\ \mathcal{E}(\mathcal{G}_d) &= \{(v_2, v_3), (v_3, v_2)\}. \end{aligned}$$

In addition, suppose that all agents detect and update their way-points synchronously and on the time intervals $[4k, 4k+1)$, $[4k+1, 4k+2)$, $[4k+2, 4k+3)$, and $[4k+3, 4(k+1))$, update actions occur $2k$, 1, $2k+1$, and 1 times, respectively, and that at each update time of agent i, $i = 1, 2, 3$, $\mathcal{D}_k^i$ is a singleton, obtained by averaging the position variables of agent i together with its neighbors. Under the above assumptions, the evolution of agents' states with respect to update times can be equivalently represented by the discrete-time system studied in section IV.C in [19]. Proposition 4 in that paper shows the states of agents with initial states $x_1(0) = 0$ and $x_2(0) = x_3(0) = 1$ do not converge to a common value as time evolves.

3.2. Network connectivity maintenance. In many cases, the interaction topology depends on the positions of all agents and thus the time-dependent condition of "periodical union of the interaction graph" is difficult to test. A trade-off approach is to add additional distributed control rules to maintain the network connectivity and thus ensure the solvability of the rendezvous problem. Related issues will be discussed in this subsection. To begin with, we first give one basic fact about the motion area of all agents.

LEMMA 3.2. *For any* $i \in \mathcal{I}_n$ *and* $t \geq 0$, $x_i(t) \in \text{co}\{x_j(t') : -\tau_d \leq t' \leq 0, j \in \mathcal{I}_n\}$.

Proof. The lemma is a direct consequence of Lemma 2.2 and assumptions (A1) through (A4). ∎

For convenience, denote

$$R_{\max} = \sup\{\|\xi - \zeta\|_2 : \xi, \zeta \in \text{co}\{x_i(t) : -\tau_d \leq t \leq 0, i \in \mathcal{I}_n\}\}.$$

Assume that each agent can detect the relative positions of all the other agents, located within the disk with radius R and with the agent as the center; in other words, mathematically, the sensing range of agent i is described by $\{\xi \in \mathbb{R}^N : \|\xi - x_i\|_2 \leq R\}$. In this case, the interaction topology $\mathcal{G}(t)$ is undirected and $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(t))$ if and only if $\|x_i(t) - x_j(t)\|_2 \leq R$.

First consider the one-dimensional case with $n \geq 2$.

Let $L = \lceil \frac{\tau_d}{\tau_{\min}} \rceil + 1$, where $\lceil \frac{\tau_d}{\tau_{\min}} \rceil$ denotes the minimum integer not less than $\frac{\tau_d}{\tau_{\min}}$, and introduce two positive real numbers S and D such that $R \geq (L+2)S + D$. The meanings of these parameters are explained in the forthcoming lemma. Moreover, this lemma also lists the principles of the selection of weighting factors and the selection of way-points in realizing network connectivity maintenance.

LEMMA 3.3. *Assume assumptions* (A1) *through* (A4) *and the following:*

(A5) $\frac{W^U}{W^L} \geq \max\{\frac{(n-1)R_{\max}}{S} - (n-1), \frac{(n-2)R_{\max}}{D}\}$.

(A6) *Each agent chooses the nearest way-point from its anticipated-way-point set at its update times, namely, for any* $i \in \mathcal{I}_n$ *and* $k \in \mathbb{N}$, $|x_i(t_k^i) - x_i(t_{k+1}^i)| = \inf\{|x_i(t_k^i) - \xi| : \xi \in \mathcal{D}_k^i\}$.

(A7) *For any* i *and* k, *agent* i *moves from its current way-point* $x_i(t_k^i)$ *to its next way-point* $x_i(t_{k+1}^i)$ *monotonically; i.e., for* t *with* $t_k^i \leq t \leq t_{k+1}^i$, *each entry of vector* $x_i(t)$ *is a (not necessarily strictly) monotonically increasing or decreasing function.*

Then we have the following facts:

(F1) *For any* i *and* k, $|x_i(t_{k+1}^i) - x_i(t_k^i)| \leq S$.

(F2) *For any* $j \in \mathcal{N}_i^A(t_k^i)$, *if* $|x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i)| \geq D$, *then for* $t_k^i \leq t \leq t_{k+1}^i$, *if* $x_j(t_k^i - \tau_k^{ij}) < x_i(t_k^i)$, *then* $x_i(t) \leq x_i(t_k^i)$, *or otherwise* $x_i(t_k^i) \leq x_i(t)$.

(F3) *For any* i, j, *and* k, *if* $t_k^i - \tau_k^{ij} \geq 0$, *then* $|x_j(t_k^i - \tau_k^{ij}) - x_j(t_k^i)| \leq LS$.

Proof. See the appendix for the proof. ∎

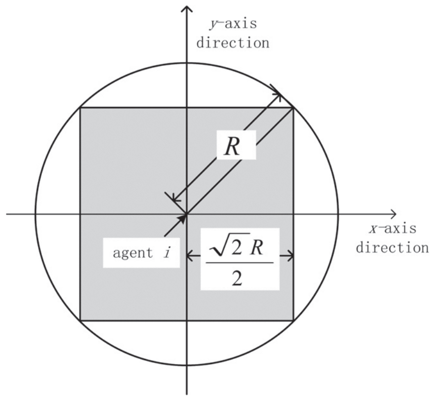

FIG. 3.1. *Revised sensing range in two-dimensional network connectivity-preserving control. The area within the circle is the original sensing range of agent i and the shaded area is the shrunk sensing range.*

In Lemma 3.3, (F1) says that the maximum step length between any two consecutive way-points is S, and (F2) says that if $j \in \mathcal{N}_i^A(t_k^i)$ and the distance between points $x_i(t_k^i)$ and $x_j(t_k^i - \tau_k^{ij})$ is not less than D, then agent i keeps stationary or moves toward point $x_j(t_k^i - \tau_k^{ij})$ after time t_k^i. Facts (F1) and(F2) play an important role in preserving the network connectivity. In the proof of Theorem 3.4, we will see that assumptions (A1) through (A4) and (A7) and facts (F1) and (F2) are also served as a sufficient condition.

The following theorem states the network connectivity-preserving property under the proposed protocols.

THEOREM 3.4 (connectivity maintenance). *Suppose that $N = 1$ and the studied system, with initial states $x_i(t) = x_i(0)$, $-\tau_d \leq t \leq 0$, $i \in \mathcal{I}_n$, satisfies assumptions* (A1) *through* (A7), *and suppose that for any $t \geq 0$, the relative positions of all neighboring agents, determined by interaction topology $\mathcal{G}(t)$, will be detected and used in the computation of way-point sets; in other words, given any $k \in \mathbb{N}$, if $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(t))$ for all $0 \leq t \leq t_k^j$, then $i \in \mathcal{N}_j^A(t_k^j)$. Then $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(0))$ implies that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(t))$ for all $t \geq 0$.*

Proof. See the appendix for the proof. □

By the above theorem, to guarantee the solvability of the rendezvous problem, we only need to ensure that the graph $\mathcal{G}(0)$ is connected. Next, we take the two-dimensional example to show how to extend the above connectivity-preserving strategy to the general case.

Assume that all agents share a global coordinate system (this assumption can be relaxed and replaced by the assumption that the axes of each agent's local coordinate system are parallel to those of others') and assume that each agent regulates its position with respect to the x-axis and the y-axis, respectively. To ensure that all agents can regulate their positions in each axis direction independently, we shrink their sensing range artificially to the square with a side length of $\sqrt{2}R$, covered by the original one; see Figure 3.1. Mathematically, the shrunk sensing range of agent i is $\{\xi \in \mathbb{R}^2 : \|\xi - x_i\|_\infty \leq \frac{\sqrt{2}R}{2}\}$.

With the above preparations, we get the following corollary.

COROLLARY 3.5. *Suppose that $N = 2$ and employ the same distributed control strategy as assumed in Theorem* 3.4, *with $\frac{\sqrt{2}R}{2}$ in the place of R, to each coordinate*

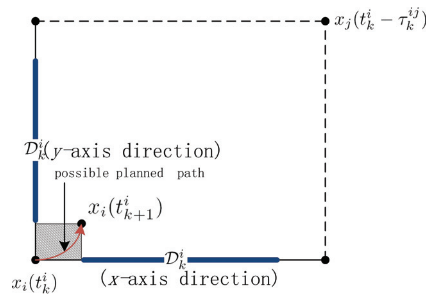

FIG. 3.2. *Allowable motion region in two-dimensional synchronous network connectivity-preserving control. The shaded area represents the allowable motion region.*

direction of the studied system. If $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(0))$, then $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(t))$ for all $t \geq 0$, where $\mathcal{G}(t)$ is determined by the shrunk sensing ranges.

Proof. See the the appendix for the proof. □

Remark. In the above example, we can add the "synchrony" requirement to the control over each coordinate direction without losing the connectivity-preserving property. Explicitly, each agent shares a common update time sequence in both x-axis and y-axis directions. In this synchronous case, the allowable motion region at time t_k^i becomes a rectangle with $x_i(t_k^i)$ and $x_i(t_{k+1}^i)$ as the diagonal vertices; see Figure 3.2, which shows the allowable motion region and possible planned path of agent i when there exists only one neighbor at update time t_k^i.

4. Technical proof. This section characterizes the set-valued consensus property of anticipated-way-point set $\mathcal{D}_k^i$ with respect to update times and proves the correctness of Theorem 3.1. In the first subsection, we manage to represent the studied nonlinear system by the equivalent dimension-augmented discrete-time system (4.9). By the preliminary lemmas assembled in the second subsection and by a dimension-reducing approach, the algebraic and graph properties of state matrix $\Xi(k)$ of discrete-time system (4.9) are investigated in the third subsection. Finally, Theorem 3.1 is proved in the last subsection.

4.1. Equivalent representation. Collect all update times t_k^i, $k \in \mathbb{N}$, $i \in \mathcal{I}_n$, and relabel them by $t_0, t_1, t_2, \dots$ in increasing order such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$.

LEMMA 4.1. *There exists a positive integer K such that for all k, $t_{k+K} - t_k \geq \max\{\tau_{\max}, \tau_d\}$.*

Proof. Let $\tau = \max\{\tau_{\max}, \tau_d\}$. By assumption (A1), for any $i \in \mathcal{I}_n$, agent i updates its anticipated-way-point set at most $\lceil \frac{\tau}{\tau_{\min}} \rceil$ times on the time interval $(t_k, t_k + \tau]$. Let $K = n\lceil \frac{\tau}{\tau_{\min}} \rceil$ and then $t_{k+K} \geq t_k + \tau$. □

Note that for any i, the anticipated-way-point sets $\mathcal{D}_k^i$, $k \in \mathcal{I}_n$, may not have their definitions on some times in the time sequence $t_0, t_1, t_2, \dots$. To determine the effective anticipated-way-point set of each agent by the new labeled time sequence, we introduce a *delay operator* $\delta : \mathcal{I}_n \times [0, \infty) \to \mathbb{N}$, defined by that for $i \in \mathcal{I}_n$ and $t \geq 0$, $t_{\delta(i,t)}^i \leq t < t_{\delta(i,t)+1}^i$. For symbolic simplicity, in the rest of this section, denote $\delta(i, t_k)$ by $\delta(i, k)$. By its definition, for $i \in \mathcal{I}_n$ and $k \in \mathbb{N}$, $t_{\delta(i,k)}^i \leq t_k < t_{\delta(i,k)+1}^i$. Clearly, it may happen that $\delta(i, k) = \delta(i, k+1)$ except when $t_{k+1} = t_{\delta(i,k+1)}^i$. From Lemma 4.1 and assumption (A1), it follows that

$$t_{\delta(i,k)}^i > t_{k-K}, \quad i \in \mathcal{I}_n. \tag{4.1}$$

For simplicity, let

$$\mathcal{Y}_i(k) = \left\{ x_i(t^i_{\delta(i,k)}), x_j(t^i_{\delta(i,k)} - \tau^{ij}_{\delta(i,k)}), j \in \mathcal{N}^A_i(t^i_{\delta(i,k)}) \right\}, i \in \mathcal{I}_n, k \in \mathbb{N}.$$

Obviously, it is not a collection of all agents' states at time t_k and the number of the elements in $\mathcal{Y}_i(k)$ is not greater than n. Just like $\mathcal{A}^i_k$, the members in $\mathcal{Y}_i(k)$ are distinguished by their symbolic expressions, but not by their numerical values. By assumption (A3) and Lemma 2.2,

$$x_i(t^i_{\delta(i,k)+1}) \in \mathcal{D}^i_{\delta(i,k)} \subset \mathrm{co}(\mathcal{Y}_i(k)). \tag{4.2}$$

Denote all elements in $\mathcal{Y}_i(k)$ by $y^i_1(k), y^i_2(k), \ldots, y^i_n(k)$, allowing repetitions, in such a way that

$$y^i_j(k) = \begin{cases} x_j(t^i_{\delta(i,k)} - \tau^{ij}_{\delta(i,k)}), & j \in \mathcal{N}^A_i(t^i_{\delta(i,k)}), \\ x_i(t^i_{\delta(i,k)}), & \text{otherwise}, \end{cases} \tag{4.3}$$

and define vectors

$$y_i(k) = [y^i_1(k)^T, y^i_2(k)^T, \ldots, y^i_n(k)^T]^T, i \in \mathcal{I}_n,$$

and

$$y(k) = [y_1(k)^T, y_2(k)^T, \ldots, y_n(k)^T]^T.$$

It follows from (2.1) that for any $\xi \in \mathcal{D}^i_{\delta(i,k)}$, there exist positive real numbers $\omega_1, \omega_2, \ldots, \omega_n$ such that

$$\xi = \frac{\sum_{j=1}^n \omega_j y^i_j(k)}{\sum_{j=1}^n \omega_j}$$

and

$$\begin{cases} W^L \le \omega_j \le W^U \text{ for } j \in \mathcal{N}^A_i(t^i_{\delta(i,k)}), \\ W^L \le \sum_{j \notin \mathcal{N}^A_i(t^i_{\delta(i,k)})} \omega_j \le W^U, \\ \omega_r = \omega_s \text{ for all } r, s \notin \mathcal{N}^A_i(t^i_{\delta(i,k)}). \end{cases} \tag{4.4}$$

It can be easily proved that all possible coefficient vectors $\frac{1}{\sum_{j=1}^n \omega_j}[\omega_1, \omega_2, \ldots, \omega_n]$, satisfying the property described by (4.4), constitute a compact set $\mathcal{W}^i_{\delta(i,k)}$. Conversely, for any $\omega \in \mathcal{W}^i_{\delta(i,k)}$, we have $(\omega \otimes I_N) y_i(k) \in \mathcal{D}^i_{\delta(i,k)}$, where $\otimes$ denotes the Kronecker product. Therefore, there exists a one-to-one correspondence between $\mathcal{D}^i_{\delta(i,k)}$ and $\mathcal{Y}_i(k)$ (or $y_i(k)$) through set $\mathcal{W}^i_{\delta(i,k)}$. Because the number of all possible cases of set $\mathcal{N}^A_i(t^i_{\delta(i,k)})$ is finite, the number of all possible sets $\mathcal{W}^i_{\delta(i,k)}$, $k \in \mathbb{N}$, is also finite. Therefore $\mathcal{W} = \bigcup_{i=1}^n \bigcup_{k=0}^\infty \mathcal{W}^i_{\delta(i,k)}$ is a compact set. To conclude, we have the next lemma.

LEMMA 4.2. *$\mathcal{W}$ is a compact set and each entry of any member in $\mathcal{W}$ is larger than 2μ, where $\mu = \frac{W^L}{2n^2 W^U}$.*

Now, we aim to give the evolution equation of variable $y(k)$ with respect to parameter k. Consider variable $y_i(k+1)$, $i \in \mathcal{I}_n$. The simplest case is that in which $\delta(i, k+1) = \delta(i, k)$, and thus

$$y_i(k+1) = y_i(k). \tag{4.5}$$

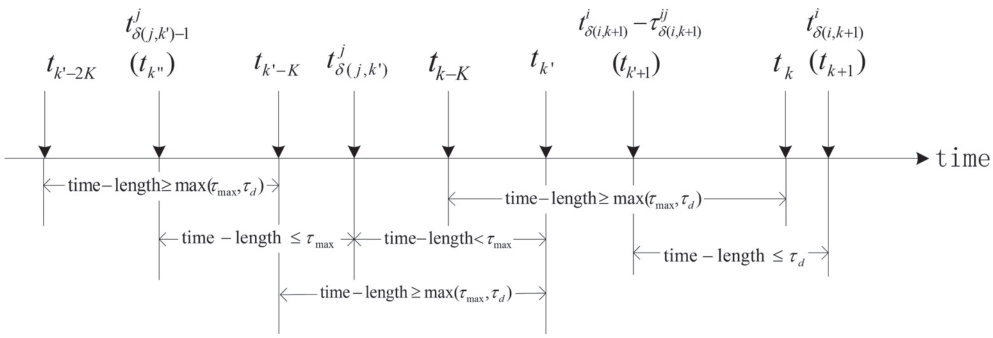

FIG. 4.1. *The range estimation of event times when* $\delta(i,k+1) = \delta(i,k)+1$.

Next, we study the case when $\delta(i,k+1) = \delta(i,k)+1$, which implies that $t_{k+1} = t^i_{\delta(i,k+1)}$.

(1) For $j \notin \mathcal{N}_i^A(t^i_{\delta(i,k+1)})$, by (4.2),

$$y^i_j(k+1) = x_i(t^i_{\delta(i,k+1)}) \in \mathcal{D}^i_{\delta(i,k)},$$

and thus there exists a row vector $\omega^{ii}(k) \in \mathcal{W}$ such that

$$y^i_j(k+1) = (\omega^{ii}(k) \otimes I_N) y_i(k). \tag{4.6}$$

(2) If $j \in \mathcal{N}_i^A(t^i_{\delta(i,k+1)})$, then $y^i_j(k+1) = x_j(t^i_{\delta(i,k+1)} - \tau^{ij}_{\delta(i,k+1)})$. Suppose that k is sufficiently large, such as $k > 3K-1$. By Lemma 4.1 and inequality (4.1), $t^i_{\delta(i,k+1)} = t_{k+1}$ and $t_{k-K} < t^i_{\delta(i,k+1)} - \tau^{ij}_{\delta(i,k+1)} \le t_{k+1}$; see Figure 4.1. Let $k-K \le k' \le k$ such that $t_{k-K} \le t_{k'} < t_{k'+1} = t^i_{\delta(i,k+1)} - \tau^{ij}_{\delta(i,k+1)}$. Then by assumption (A4),

$$y^i_j(k+1) \in \mathrm{co}(\mathcal{D}^j_{\delta(j,k')} \cup \mathcal{D}^j_{\delta(j,k')-1}). \tag{4.7}$$

Also by inequality (4.1), $t^j_{\delta(j,k')} > t_{k'-K}$, which further implies that $t^j_{\delta(j,k')-1} > t_{k'-2K}$. Let $t_{k''} = t^j_{\delta(j,k')-1}$. Then the preceding inequality implies that $k'' > k'-2K$. Let $m = 3K-1$. Then $k-m \le k'' < k' \le k$, and by (4.7),

$$y^i_j(k+1) \in \mathrm{co}(\mathcal{D}^j_{\delta(j,k')} \cup \mathcal{D}^j_{\delta(j,k'')}) \subset \mathrm{co}(\mathcal{Y}_j(k') \cup \mathcal{Y}_j(k'')).$$

Thus there exists vectors $\omega^{ij1}, \omega^{ij2} \in \mathcal{W}$ and real number $\alpha \in [0,1]$ such that

$$y^i_j(k+1) = \alpha(\omega^{ij1}(k) \otimes I_N) y_j(k') + (1-\alpha)(\omega^{ij2}(k) \otimes I_N) y_j(k''). \tag{4.8}$$

Note that in the above equation, the selection of k' and k'' uniquely depends on parameters i, j, k, and the selection of α depends on vectors $\omega^{ij1}(k)$ and $\omega^{ij2}(k)$.

Define state variable $z(k) = [y(k)^T, y(k-1)^T, \ldots, y(k-m)^T]^T$ and define matrix

$$\Xi(k) = \begin{bmatrix} A_0(k) & A_1(k) & \cdots & A_{m-1}(k) & A_m(k) \\ I_{nn} & & & & \\ & I_{nn} & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & I_{nn} & 0 \end{bmatrix},$$

where $k > m$, I_{nn} is the $nn \times nn$ identity matrix, $A_l(k) = [A_{ij}^l(k)]$, $l = 0, 1, 2, \ldots, m$, are $n \times n$ block matrices, and blocks $A_{ij}^l \in \mathbb{R}^{n\times n}$ are determined in the following way that for any $i \in \mathcal{I}_n$,

(1) If $\delta(i, k+1) = \delta(i, k)$, then $A_{ii}^0(k) = I_n$, and matrices $A_{ij}^0(k)$, $j \in \mathcal{I}_n$, $j \neq i$, and $A_{ij}^l(k)$, $l = 1, 2, \ldots, m$, $j \in \mathcal{I}_n$, are all zero matrices;

(2) If $\delta(i, k+1) = \delta(i, k) + 1$, then for $j = 1, 2, \ldots, n$,

(2.1) If $j \notin \mathcal{N}_i^A(t^i_{\delta(i,k+1)})$, then the jth row of $A_{ii}^0(k)$ equals $\omega_{ii}(k)$, decided by (4.6);

(2.2) If $j \in \mathcal{N}_i^A(t^i_{\delta(i,k+1)})$, then the jth rows of $A_{ij}^{k-k'}$ and $A_{ij}^{k-k''}$ equal $\alpha\omega^{ij1}$ and $(1-\alpha)\omega^{ij2}$, respectively, where k', k'', $\omega^{ij1}(k)$, $\omega^{ij2}(k)$ are given by (4.8);

(2.3) The jth rows of all other matrices in $\{A_{ir}^l,\ l = 0, 1, \ldots, m,\ r \in \mathcal{I}_n\}$, which are not defined in steps (2.1) and (2.2), equal zeros.

Clearly, $\Xi(k)$ is a *stochastic* matrix, namely, $\Xi(k)$ is a square matrix with the property that all its row sums are 1.

Combining (4.5), (4.6), and (4.8), the evolution process of the anticipated-way-point sets can be simply represented by the following discrete-time equation:

$$z(k+1) = (\Xi(k) \otimes I_N) z(k), \quad k > m. \tag{4.9}$$

The next lemma presents an equivalent statement of Theorem 3.1.

LEMMA 4.3. *Under the assumptions assumed by Theorem* 3.1, *the rendezvous problem is solvable if and only if there exists a column vector* $x^* \in \mathbb{R}^N$ *such that*

$$\lim_{k\to\infty} z(k) = \mathbf{1} \otimes x^*, \tag{4.10}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$ *with compatible dimension.*

Proof. The necessity is a direct consequence of the definition of vector $z(k)$ and we only prove the sufficiency. Obviously, (4.10) implies that for any i, $\lim_{k\to\infty} y_i(k) = \mathbf{1} \otimes x^*$, where $\mathbf{1} \in \mathbb{R}^n$. For any given $\varepsilon > 0$, by (4.2) there exists $k^* \in \mathbb{N}$ such that for any $k \geq k^*$ and any i, $\mathcal{D}_k^i \subset \mathcal{B}(x^*, \varepsilon)$. Therefore, by assumption (A4), $\lim_{t\to\infty} x_i(t) = x^*$ for any i, namely, all agents will reach a common location x^* asymptotically. □

The assumption on the interaction topology assumed by Theorem 3.1 is restated by Lemma 4.4.

LEMMA 4.4 (cf. [23, Lemma 9]). *The existence of* $T \geq 0$ *such that for all* $t^0 \geq 0$, *the union of graph* $\mathcal{G}^A(t)$ *across the time interval* $[t^0, t^0 + T]$ *always contains a spanning tree, is equivalent to the condition that there exists a positive integer* K_T *with the property that for any* k *the union of* $\mathcal{G}^A(t)$ *on* $\{t_k, t_{k+1}, \ldots, t_{k+K_T}\}$ *contains a spanning tree.*

Proof. Necessity: Let $K_T = n\lceil \frac{T}{\tau_{\min}} \rceil$. By the same arguments as in Lemma 4.1, we get that $t_{k+T_K} \geq t_k + T$. Because $\mathcal{G}^A(t)$ is constant on $[t_k, t_{k+1})$, $k \in \mathbb{N}$, the union of $\mathcal{G}^A(t)$ on $\{t_k, t_{k+1}, \ldots, t_{k+K_T}\}$ contains a spanning tree.

Sufficiency: For any t^0, there exists k such that $t_k \leq t^0 < t_{k+1}$. By assumption (A1), $t_{l+1} - t_l \leq \tau_{\max}$ for any $l \in \mathbb{N}$, which means that $t_{k+1+K_T} \leq t_k + (K_T + 1)\tau_{\max} \leq t^0 + (K_T+1)\tau_{\max}$. Let $T = (K_T+1)\tau_{\max}$. Then $\{t_{k+1}, t_{k+2}, \ldots, \ldots, t_{k+1+K_T}\} \subset [t^0, t^0 + T]$ and thus the sufficient part holds. □

To sum up, in this subsection, for any $i \in \mathcal{I}_n$, we first found a nN vector $y_i(k)$, which uniquely corresponds to anticipated-way-point set $\mathcal{D}^i_{\delta(i,k)}$, and then we transformed the set-valued consensus into its equivalent dimension-augmented discrete-time consensus representation, described by (4.10). The special structures of state matrix $\Xi(k)$ will be further studied in subsection 4.3. Finally, in Lemma 4.4, we gave the equivalent statement of Theorem 3.1.

4.2. Preliminary lemmas. This subsection lists some preliminary notions and lemmas that are needed to show the convergence property of discrete-time system (4.9). The matrix notions employed in this subsection have their independent meanings and are not the ones defined elsewhere.

A *weighted* directed graph $\mathcal{G}(A)$ is a directed graph $\mathcal{G}$ plus a nonnegative weight matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff a_{ji} > 0$. A stochastic matrix A is called indecomposable and aperiodic (SIA) if there exists a column vector ν such that $\lim_{k\to\infty} A^k = \mathbf{1}\nu^T$. In what follows, let $\prod_{i=1}^k A_i = A_k A_{k-1}\cdots A_1$, denoting the left product of matrices, and write $A \geqslant B$ if $A - B$ is nonnegative.

The following two lemmas give sufficient conditions ensuring that the product of a set of SIA matrices converges and ensuring that a stochastic matrix is SIA, respectively.

Lemma 4.5 (see [20, Lemma 5]). *Let $\mathcal{A}$ be a compact set consisting of SIA matrices with the same dimension and with the property that for any nonnegative integer k and any $A_1, A_2, \ldots, A_k \in \mathcal{A}$ (repetitions permitted), $\prod_{i=1}^k A_i$ is SIA. Then given any infinite sequence $A_1, A_2, A_3, \ldots$ (repetitions permitted) of matrices in $\mathcal{A}$, there exists a column vector ν such that $\lim_{k\to\infty}\prod_{i=1}^k A_i = \mathbf{1}\nu^T$.*

Lemma 4.6 (see [28, Lemma 1]). *Let A be a stochastic matrix. If $\mathcal{G}(A)$ has a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(A)$, then A is SIA.*

The following lemma is useful in building the connection between the interaction topology $\mathcal{G}^A(t)$ and the matrix $\Xi(k)$ related to system (4.9).

Lemma 4.7 (see [23, Lemma 8]). *Let $A_0, A_1, \ldots, A_m$ be $n\times n$ nonnegative matrices, let*

$$D = \begin{bmatrix} A_0 & A_1 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+1)n\times(m+1)n},$$

let

$$M_0 = \begin{bmatrix} I_n & & & & \\ I_n & & & 0 & \\ & I_n & & & \\ & & \ddots & & \\ 0 & & & I_n & 0 \end{bmatrix}_{(m+1)n\times(m+1)n},$$

and let $M_k = D + {M_0}^k$ for any $k \in \{1, 2, \ldots, m\}$. Then if $\mathcal{G}(\sum_{i=1}^m A_i)$ contains a spanning tree, $\mathcal{G}(M_k)$ contains a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(M_k)$. Furthermore, the two root vertices in the associated two spanning trees have the same index label.

4.3. Properties of matrix $\Xi(k)$. Let $\mathcal{M}$ denote the matrix set, constituted by all the possible matrices with the form

$$B = \begin{bmatrix} B_0 & B_1 & \cdots & B_{m-1} & B_m \\ I_{nn} & & & & \\ & I_{nn} & & 0 & \\ & & \ddots & & \\ & 0 & & I_{nn} & 0 \end{bmatrix}, \tag{4.11}$$

where $B_k = [B_{ij}^k]$, $k = 0, 1, \ldots, m$, are $n \times n$ block matrices and blocks $B_{ij}^k \in \mathbb{R}^{n\times n}$ satisfy the property that for any $i \in \mathcal{I}_n$,

(1) If B_{ii}^0 is an identity matrix, then matrices B_{ij}^0, $j \in \mathcal{I}_n$, $j \neq i$, and matrices B_{ij}^k, $j \in \mathcal{I}_n$, $k = 1, 2, \ldots, m$, are all zero matrices;

(2) Otherwise,

(2.1) The ith row of B_{ii}^0 belongs to compact set $\mathcal{W}$ and the ith rows of all other matrices B_{ij}^0, $j \in \mathcal{I}_n$, $j \neq i$, B_{ij}^k, $j \in \mathcal{I}_n$, $k = 1, 2, \ldots, m$, are zeros;

(2.2) For $j = 1, 2, \ldots, n$, $j \neq i$, there exists $\omega \in \mathcal{W}$ such that the jth row of B_{ii}^0 equals ω or there exist $0 \leq k_1 < k_2 \leq m$, $\omega^1, \omega^2 \in \mathcal{W}$, and $\alpha \in [0, 1]$ such that the jth rows of $B_{ij}^{k_1}$ and $B_{ij}^{k_2}$ equal $\alpha\omega^1$ and $(1-\alpha)\omega^2$, respectively; the jth rows of all other matrices are zeros.

For convenience, let $\pi(B_0, B_1, \ldots, B_m)$ denote matrix B with the form given by (4.11).

By the above definition, we have the next lemma.

LEMMA 4.8. *Matrix set $\mathcal{M}$ is a compact set and includes all possible matrices $\Xi(k)$, namely, $\Xi(k) \in \mathcal{M}$ for all $k > m$.*

To characterize matrix set $\mathcal{M}$ further, introduce n matrices $N_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}_n$, which possess the property similar to identity matrix and are defined by that the ith row of N_i are all 1 and all other rows are zeros. Then

$$N_i N_j = N_i \text{ for all } i, j \in \mathcal{I}_n. \tag{4.12}$$

The properties of the members in $\mathcal{M}$ are summarized by the following lemma.

LEMMA 4.9. *For any $B = \pi(B_0, B_1, \ldots, B_m) \in \mathcal{M}$, we have the following:*

(1) For any i, if $B_{ii}^0 \neq I_n$, then $B_{ii}^0 > 2\mu N_i$, where $\mu = \frac{W^L}{2n^2 W^U}$ is defined in Lemma 4.2.

(2) For any i and j, $i \neq j$, $B_{ii}^0 > 2\mu N_j$ or there exists $l \in \{0, 1, 2, \ldots, m\}$ such that $B_{ij}^l \geq \mu N_j$.

Let $\epsilon = \max\{K + m, K_T + 1\}$, where K and K_T are defined in Lemma 4.1 and Lemma 4.4, respectively. Consider the product of a sequence of finite matrices $\pi(B_0^k, B_1^k, \ldots, B_m^k) \in \mathcal{M}$, $k = 1, 2, \ldots, \epsilon$ (repetitions permitted). Denote matrices B_l^k, $l = 0, 1, \ldots, m$, $k = 1, 2, \ldots, \epsilon$, by $n \times n$ block matrices $[B_{ij}^{kl}]$, where blocks $B_{ij}^{kl} \in \mathbb{R}^{n\times n}$, and denote matrix $\prod_{k=1}^{\epsilon} \pi(B_0^k, B_1^k, \ldots, B_m^k)$ by $(m+1)n \times (m+1)n$ block matrix $D = [D_{ij}]$, where blocks $D_{ij} \in \mathbb{R}^{n\times n}$. Then

(3) If for any $i \in \mathcal{I}_n$, B_{ii}^{k0}, $k = 1, 2, \ldots, \epsilon - m$, are not all identity matrices, then for any r, $1 \leq r \leq (m+1)n$, and $s \in \mathcal{I}_n$, there exists l, $1 \leq l \leq (m+1)n$, such that $D_{rl} \geq \mu^{\epsilon} N_s$.

Proof. The first two properties follow directly from the definition of matrix set $\mathcal{M}$ and we prove only the last one.

Denote the matrix $\prod_{l=1}^{k} \pi(B_0^l, B_1^l, \dots, B_m^l)$ by $(m+1)n \times (m+1)n$ block matrix $P_k = [P_{ij}^k]$, where $P_{ij}^k \in \mathbb{R}^{n\times n}$, and let $\mathrm{Ndiag}(B_0^k)$ represent the $n \times n$ diagonal block matrix, defined by that for any $i \in \mathcal{I}_n$, if $B_{ii}^{k0} \neq I_n$, then the ith diagonal block equals N_i; otherwise the ith diagonal block equals I_n. Then for any k,

$$\pi(B_0^k, B_1^k, \dots, B_n^k) \geq \mu \begin{bmatrix} \mathrm{Ndiag}(B_0^k) & & & \\ I_{nn} & & \text{\Large 0} & \\ & & \ddots & \\ \text{\Large 0} & & I_{nn} & 0 \end{bmatrix}.$$

First consider the case with $r \in \mathcal{I}_n$. Let $k_1, k_2, \dots, k_p$ be the index such that $B_{rr}^{k_i 0} \neq I_n$, $i = 1, 2, \dots, p$, and let $k_{p+1} = \epsilon + 1$. Clearly, $k_1 \leq \epsilon - m$. Fix r, s, and index i, $i \leq p$. By statement (2), $B_{rr}^{k_i 0} > 2\mu N_s$ or there exists $l \in \{0, 1, 2, \dots, m\}$ such that $B_{rs}^{k_i l} \geq \mu N_s$. Since for any k, $k_i < k < k_{i+1}$, the rth diagonal block of B_0^k is the $n \times n$ identity matrix, we get that for any k, $k_i \leq k < k_{i+1}$,

$$\begin{aligned}[P_{r1}^k, P_{r2}^k, \dots, P_{r,(m+1)n}^k] \geq{}& \mu^{k_i - 1}[B_{r1}^{k_i 0}, B_{r2}^{k_i 0}, \dots, B_{rn}^{k_i 0}, B_{r1}^{k_i 1}, B_{r2}^{k_i 1}, \dots, B_{rn}^{k_i 1}, \\ & \dots\dots\dots, B_{r1}^{k_i m}, B_{r2}^{k_i m}, \dots, B_{rn}^{k_i m}] \\ & \times \prod_{l=1}^{k_i - 1} \begin{bmatrix} \mathrm{Ndiag}(B_0^l) & & & \\ I_{nn} & & \text{\Large 0} & \\ \text{\Large 0} & & \ddots & \\ & & I_{nn} & 0 \end{bmatrix}.\end{aligned}$$

In the case that $B_{rr}^{k_i 0} > 2\mu N_s$, it can be obtained that $P_{rr}^k \geq \mu^{k_i} N_s$. Consider the other case, that is, there exists $l \in \{0, 1, 2, \dots, m\}$ such that $B_{rs}^{k_i l} \geq \mu N_s$. If $l = 0$, then by the above inequality, $P_{rs}^k \geq \mu^{k_i} N_s$. Otherwise, $P_{(l-1)n+r,s}^k \geq \mu^{k_i} N_s$. Noticing that $\mu < 1$, we have the conclusion that for any $r, s \in \mathcal{I}_n$ and any k, $\epsilon - m \leq k \leq \epsilon$, there exists l, $1 \leq l \leq (m+1)n$, such that $P_{rl}^k \geq \mu^{\epsilon} N_s$.

Since the matrix in the lower-left corner of $\pi(B_0^k, B_1^k, \dots, B_n^k)$ is an $mnn \times mnn$ identity matrix, we can relax the assumption that $r \in \mathcal{I}_n$ and get statement (3). $\square$

To establish a connection between $\mathcal{G}^A(t_{k+1})$ and $\Xi(k)$, define a *dimension-reducing* map from $\mathcal{M}$ to $\mathbb{R}^{(m+1)n\times(m+1)n}$. For each matrix B in $\mathcal{M}$ with the form given by (4.11), its image under the map is denoted by $\hat{B}$ and has the form that

$$\hat{B} = \begin{bmatrix} \hat{B}_0 & \hat{B}_1 & \cdots & \hat{B}_{m-1} & \hat{B}_m \\ I_n & & & & \\ & I_n & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & I_n & 0 \end{bmatrix},$$

where $\hat{B}_k = [b_{ij}^k]$, $k = 0, 1, \dots, m$, are $n \times n$ matrices defined by

$$b_{ij}^k = \begin{cases} 1 & \text{if } B_{ij}^k = I_n \text{ (which happens only when } k = 0 \text{ and } i = j) \\ & \text{or there exists some } s \in \mathcal{I}_n \text{ such that } B_{ij}^k \geq \mu N_s, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 4.9(1) that $\hat{B}_0 \geq I_n$. For simplicity, denote the matrix $\hat{B}$, given above, by $\pi(\hat{B}_0, \hat{B}_1, \dots, \hat{B}_m)$.

Adopt the same notation as in Lemma 4.9, denote the graphs $\mathcal{G}(\sum_{k=1}^{\epsilon}\sum_{i=0}^{m}\hat{B}_i^k)$, $\mathcal{G}(\prod_{k=1}^{\epsilon}\pi(\hat{B}_0^k,\hat{B}_1^k,\ldots,\hat{B}_m^k))$, and $\mathcal{G}(\prod_{k=1}^{\epsilon}\pi(B_0^k,B_1^k,\ldots,B_m^k))$ by $\mathcal{G}_v$, $\mathcal{G}_u$ and $\mathcal{G}_u^*$, respectively, and denote their vertex sets by $\{v_1,v_2,\ldots,v_n\}$, $\{u_1,u_2,\ldots,u_{(m+1)n}\}$, and

$$\left\{u_1^1,u_2^1,\ldots,u_n^1;u_1^2,u_2^2,\ldots,u_n^2;\ldots\ldots\ldots;u_1^{(m+1)n},u_2^{(m+1)n},\ldots,u_n^{(m+1)n}\right\},$$

respectively, where the vertices in the last set are in their natural index order as in the definition of weighted directed graph, that is, if the product $\prod_{k=1}^{\epsilon}\pi(B_0^k,B_1^k,\ldots,B_m^k)$ is denoted by $[a_{ij}]$, then $(u_r^i,u_s^j)\in\mathcal{E}(\mathcal{G}_u^*)\iff a_{(j-1)n+s,(i-1)n+r}>0$.

The relationship among $\mathcal{G}_v$, $\mathcal{G}_u$, and $\mathcal{G}_u^*$ is characterized by the following lemma.

LEMMA 4.10. *If $\mathcal{G}_v$ has a spanning tree with root vertex v_l, $l\in\mathcal{I}_n$, then $\mathcal{G}_u$ has a spanning tree with root vertex u_l having a self-loop; if for any $i\in\mathcal{I}_n$, B_{ii}^{k0}, $k=1,2,\ldots,\epsilon-m$, are not all identity matrices, then under the above assumption $\mathcal{G}_u^*$ also has a spanning tree with root vertex u_l^l having a self-loop.*

Proof. Step 1. The first part follows from Lemma 4.7 by the same arguments as in proving Lemma 11 in [23]. To make the paper self-contained, we give a sketch of the proof. Let M_0 be the same as in Lemma 4.7 and let

$$D_k=\begin{bmatrix}\hat{B}_0^k & \hat{B}_1^k & \cdots & \hat{B}_m^k\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & 0\end{bmatrix}.$$

Then

$$\begin{aligned}\prod_{k=1}^{\epsilon}\pi(\hat{B}_0^k,\hat{B}_1^k,\ldots,\hat{B}_m^k) &\geq \frac{1}{2^\epsilon}\prod_{k=1}^{\epsilon}(M_0+D_k)\\ &\geq \frac{1}{2^\epsilon}\left[{M_0}^{\epsilon}+\sum_{k=1}^{\epsilon}{M_0}^{k-1}D_k{M_0}^{\epsilon-k}\right]\\ &\geq \frac{1}{2^\epsilon}\left[{M_0}^{\epsilon}+\sum_{k=1}^{\epsilon}D_k{M_0}^{\epsilon-k}\right].\end{aligned}\tag{4.13}$$

Let the first n rows of $D_k{M_0}^{\epsilon-k}$ be $[F_0^k,F_1^k,\ldots,F_m^k]$, where $F_i^k\in\mathbb{R}^{n\times n}$, $i=0,1,\ldots,m$. Then $\sum_{i=0}^{m}F_i^k=\sum_{i=0}^{m}\hat{B}_i^k$ and thus $\sum_{k=1}^{\epsilon}\sum_{i=0}^{m}F_i^k=\sum_{k=1}^{\epsilon}\sum_{i=0}^{m}\hat{B}_i^k$. Because $\mathcal{G}_v$ has a spanning tree with root vertex v_l, by Lemma 4.7 and inequality (4.13) and noticing that ${M_0}^{\epsilon}={M_0}^{m}$ for $\epsilon\geq m$, we have that $\mathcal{G}({M_0}^{\epsilon}+\sum_{k=1}^{\epsilon}D_k{M_0}^{\epsilon-k})$ and thus $\mathcal{G}_u$ have a spanning tree, where the associated root vertex of the spanning tree is u_l with a self-loop.

Step 2. Next, we investigate the properties of the edge set $\mathcal{E}(\mathcal{G}_u^*)$.

Suppose that $(u_j,u_i)\in\mathcal{E}(\mathcal{G}_u)$. Adopt the same notation as in proving Lemma 4.9. By (4.12) and the definition of $\mathcal{M}$, the product of finite nonzero matrices in $\{B_{rs}^{kp}: r,s\in\mathcal{I}_n,p=0,1,\ldots,m,k=1,2,\ldots,\epsilon\}$ is also nonzero, which implies that $P_{ij}^{\epsilon}\neq 0$. Moreover, it also can be obtained that $P_{ij}^{\epsilon}=I_n$ or there exists $s\in\mathcal{I}_n$ such that $P_{ij}^{\epsilon}\geq\mu^{\epsilon}N_s$. The two possible cases imply that, in $\mathcal{G}_u^*$, there exist the edges (see Figure 4.2):

$$\left\{(u_1^j,u_1^i),(u_2^j,u_2^i),\ldots,(u_n^j,u_n^i)\right\}\subset\mathcal{E}(\mathcal{G}_u^*)$$

or

$$\left\{(u_1^j,u_s^i),(u_2^j,u_s^i),\ldots,(u_n^j,u_s^i)\right\}\subset\mathcal{E}(\mathcal{G}_u^*).$$

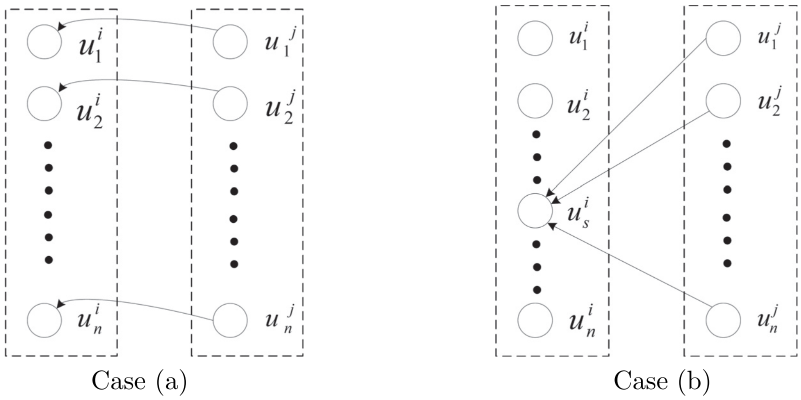

FIG. 4.2. *Two possible cases of the edges in* $\mathcal{G}_u^*$, *implied by* $(u_j, u_i) \in \mathcal{E}(\mathcal{G}_u)$.

For any $i \in \{1, 2, \ldots, (m+1)n\}$, by the conclusion of the first step, there exists a path $v_{i_1} = v_l, v_{i_2}, \ldots, v_{i_p} = v_i$, connecting u_l to u_i. Repeating the preceding arguments yields that for any $k \in \{1, 2, \ldots, p-1\}$,

$$\left\{(u_1^{i_k}, u_1^{i_{k+1}}), (u_2^{i_k}, u_2^{i_{k+1}}), \ldots, (u_n^{i_k}, u_n^{i_{k+1}})\right\} \subset \mathcal{E}(\mathcal{G}_u^*),$$

or

$$\left\{(u_1^{i_k}, u_{s_k}^{i_{k+1}}), (u_2^{i_k}, u_{s_k}^{i_{k+1}}), \ldots, (u_n^{i_k}, u_{s_k}^{i_{k+1}})\right\} \subset \mathcal{E}(\mathcal{G}_u^*) \text{ for some } s_k \in \mathcal{I}_n.$$

The above discussion implies that there exists at least one path connecting u_l^l to u_r^i for some $r \in \mathcal{I}_n$. To summarize, we have the following result.

Claim. For any $i \in \{1, 2, \ldots, (m+1)n\}$, there exists at least one path connecting the vertex u_l^l to some vertex in $\{u_r^i : r \in \mathcal{I}_n\}$.

Continue with the above deduction and let i be fixed. For any $r \in \mathcal{I}_n$, by Lemma 4.9, there exists $s \in \{1, 2, \ldots, (m+1)n\}$ such that $D_{i,s} \geq \mu^\epsilon N_r$. Thus, we have

$$\left\{(u_1^s, u_r^i), (u_2^s, u_r^i), \ldots, (u_n^s, u_r^i)\right\} \subset \mathcal{E}(\mathcal{G}_u^*).$$

By the above claim, there exists some path connecting the vertex u_l^l to some vertex in $\{u_r^s, r \in \mathcal{I}_n\}$, and thus there exists at least one path connecting the vertex u_l^l to u_r^i. By the arbitrariness of i and r, $\mathcal{G}_u^*$ has a spanning tree with root vertex u_l^l. Moreover, by Lemma 4.9(1), $D_{ll} \geq \mu^\epsilon N_l$, and thus u_l^l has a self-loop. □

Define set

$$\Pi = \Bigg\{ \prod_{k=1}^{\epsilon} \pi(B_0^k, B_1^k, \ldots, B_m^k) : \pi(B_0^k, B_1^k, \ldots, B_m^k) \in \mathcal{M}, \mathcal{G}\left(\sum_{k=1}^{\epsilon}\sum_{i=0}^{m} \hat{B}_i^k\right) \text{ has a spanning tree and for any } i \in \mathcal{I}_n,\ B_{ii}^{k0},\ k = 1, 2, \ldots, \epsilon - m, \text{ are not all identity matrices} \Bigg\}.$$

LEMMA 4.11.
(1) *Matrix set* Π *is a compact set.*
(2) *Any matrix in* Π *is SIA, and furthermore, the product of any finite matrices (repetitions permitted) in* Π *is SIA.*
(3) *The product of* ϵ *consecutive matrices* $\Xi(k)$*,* $k > m$*, belongs to* Π.

Proof. The compactness of Π follows from the definition of $\mathcal{M}$ and Lemma 4.2. The proof is trivial and details are omitted.

By Lemma 4.10 and Lemma 4.6, any matrix in Π is SIA, and following the same arguments as in proving Lemma 4.10, we can get that the product of any finite matrices (repetitions permitted) in Π is SIA.

By the definitions of $\mathcal{G}^A(t)$ and $\Xi(k)$, for any k, $\mathcal{E}(\mathcal{G}^A(t_{k+1})) \subset \mathcal{E}(\mathcal{G}(\sum_{i=0}^{m} \hat{A}_i(k)))$, and by Lemma 4.4, $\mathcal{G}(\sum_{j=k}^{k+\epsilon-1} \sum_{i=0}^{m} \hat{A}_i(j))$ has a spanning tree. In addition, since $\epsilon - m \geq K$, by Lemma 4.1 and the definition of $\Xi(k)$, $A_{ii}^0(j)$, $j = k, k+1, \ldots, k+\epsilon-m-1$, are not all identity matrices. Therefore, the last statement holds. ∎

4.4. Proof of Theorem 3.1. Let $\pi_k = \prod_{i=m+k\epsilon+1}^{m+(k+1)\epsilon} \Xi(i)$, $k \in \mathbb{N}$. Then by Lemma 4.5 and Lemma 4.11, there exists $\nu \in \mathbb{R}^{(m+1)nn}$ such that $\lim_{k\to\infty} \prod_{i=0}^{k} \pi_i = \mathbf{1}\nu^T$. For any $k > m$, there exists $l \in \mathbb{N}$ such that $m + l\epsilon + 1 \leq k \leq m + (l+1)\epsilon$. If $l \geq 1$, then

$$\prod_{i=m+1}^{k} \Xi(i) - \mathbf{1}\nu^T = \left(\prod_{i=m+l\epsilon+1}^{k} \Xi(i) \right) \left(\prod_{i=0}^{l-1} \pi_i - \mathbf{1}\nu^T \right),$$

which implies that

$$\lim_{k\to\infty} \prod_{i=m+1}^{k} \Xi(i) = \mathbf{1}\nu^T.$$

Therefore, Theorem 3.1 holds.

Remark. Following the same arguments as in the process of proving Theorem 3.1, we can relax further the sufficient conditions stated in Theorem 3.1. One simple extension is to ensure that for any time t, there exist $t_1, t_2 > t$ such that $t_2 - t_1 \geq (2^{\kappa\kappa}+1)\epsilon\tau_{\max}$, and the union of the interaction topology $\mathcal{G}^A(t)$ cross any time interval $[t^0, t^0 + T] \subset [t_1, t_2]$ contains a spanning tree, where κ is the dimension of the matrix $\Xi(k)$ in (4.9) and ϵ is given in Lemma 4.9. To demonstrate this fact, the only thing we should do is to generalize Lemma 4.5 accordingly; see [20] for its detailed proof.

Remark. If there exist measurement errors in the process of data detection, we can show that if the measurement errors are bounded by $e_{\max}$, then there exists $K_e > 0$ such that $\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_\infty \leq K_e e_{\max}$ for any i, j. The correctness of this result relies on the consensus property of discrete-time system (4.9) and the properties of scrambling matrices. A different but equivalent result in the context of connectivity preservation is presented in another paper, and its proof is available upon request.

5. Simulations. This section presents simulations to show the effectiveness of the theoretical result.

In the first example, we set $\tau_{\min} = 1$, $\tau_{\max} = 2$, $\tau_d = 4$, $W_L = 1$, and $W_U = 2$. The graph consisting of all possible information channels is depicted in Figure 5.1.

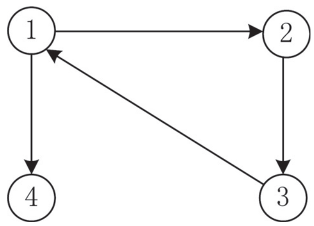

FIG. 5.1. *The graph consisting of all possible information channels in the first simulation.*

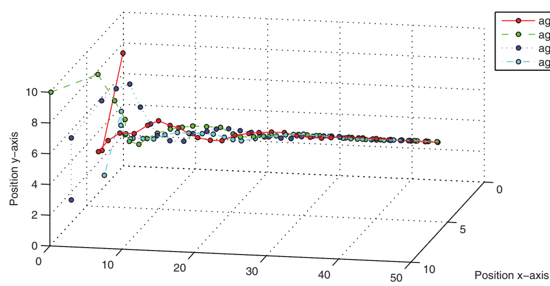

FIG. 5.2. *The evolution of way-points in the first simulation.*

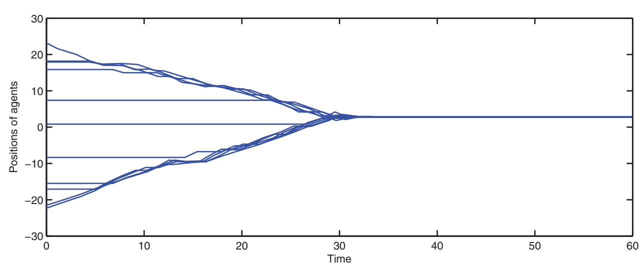

FIG. 5.3. *The trajectories of agents under the connectivity-preserving rules* (A5)–(A7).

Assume that each agent detects the positions of its adjacent agents (determined by the graph in Figure 5.1) at the times randomly distributed between its update times, and assume that the information channels may fail to work and the maximum sum of consecutive detection failures cannot exceed 4. Therefore, the interaction topology $\mathcal{G}^A(t)$ is time-varying and it satisfies the assumption of Theorem 3.1 with $T = 5\tau_{\max}$. Furthermore, assume that the detected data are randomly distributed in the region determined by assumption (A4). Figure 5.2 shows the evolution of way-points of each agent under assumptions (A1) through (A4).

To show the effectiveness of connectivity-preserving rules (A5) through (A7), we consider the system of 11 agents with the initial states as illustrated in Figure 5.3. In this example, we set $\tau_{\min} = 1$, $\tau_{\max} = 2$, $R = 10$ and choose $W_U = 315$, $W_L = 1$. It is further assumed that detections occur between update times for each agent; that is, $\tau_d = 2$. Figure 5.3 gives the trajectories of agents under assumptions (A1) through (A7) with $L = 3$, $S = 1.6$, and $D = 2$. As a comparison, Figure 5.4 gives the trajectories of agents with randomly selected weighting factors.

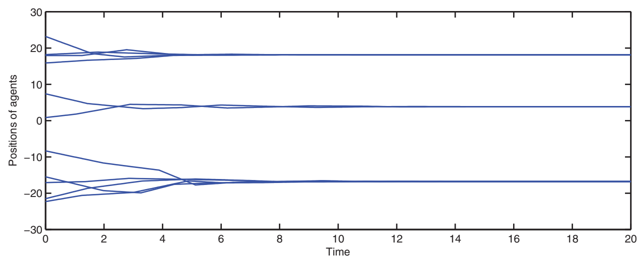

FIG. 5.4. *The trajectories of agents with randomly selected weighting factors.*

6. Conclusion. We have studied the asynchronous rendezvous problem of networks of multiple dynamic agents and presented a class of protocol-designing strategies based on weak assumptions. The nonlinear rendezvous model was treated by an equivalent set-valued consensus model, which was successfully converted into another consensus model in the traditional sense. However, its state variables are with higher dimension and the state matrices are also with some new special structures, which were not considered previously. By employing the tools of graph theory and nonnegative matrix theory, we gave its convergence proof. As an example, distributed network connectivity-preserving control was also discussed. Nevertheless, there still exist some interesting but unsolved problems, such as whether the convergence conditions can be further relaxed in the bidirectional interaction case and how to combine obstacle-avoidance algorithm to realize connectivity-preserving control. These issues are currently under investigation.

7. Appendix. *Proof of Lemma* 2.2. The compactness of $\mathcal{D}_k^i$ follows from the compactness of allowable parameter interval $[W^L, W^U]$ and the last conclusion follows from (2.1). The remaining part of the proof will show the convex property of $\mathcal{D}_k^i$. For simplicity, we omit the time parameters of state variables, which are self-evident from the context.

Suppose that $\xi = x_i + \frac{\sum_{j\in\mathcal{N}_i^A(t_k^i)} \alpha_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \alpha_l}(x_j - x_i)$, $\zeta = x_i + \frac{\sum_{j\in\mathcal{N}_i^A(t_k^i)} \beta_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \beta_l}(x_j - x_i)$ $\in \mathcal{D}_k^i$, where $W^L \le \alpha_j, \beta_j \le W^U$, $j \in \mathcal{N}_i^A(t_k^i) \cup \{i\}$. To prove $\mathcal{D}_k^i$ is convex, it suffices to prove that $\varsigma = a\xi + (1-a)\zeta \in \mathcal{D}_k^i$ for any $a \in [0, 1]$. Expanding the preceding expression of ς yields that

$$\begin{aligned}
\varsigma &= x_i + \frac{a\sum_{j\in\mathcal{N}_i^A(t_k^i)} \alpha_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \alpha_l}(x_j - x_i) + \frac{(1-a)\sum_{j\in\mathcal{N}_i^A(t_k^i)} \beta_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \beta_l}(x_j - x_i)\\
&= x_i + \sum_{j\in\mathcal{N}_i^A(t_k^i)} \left(\frac{a\alpha_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \alpha_l} + \frac{(1-a)\beta_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \beta_l}\right)(x_j - x_i)\\
&= x_i + \frac{\sum_{j\in\mathcal{N}_i^A(t_k^i)} \gamma_j}{\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \gamma_l}(x_j - x_i),
\end{aligned}$$

where

$$\gamma_j = \frac{a\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \beta_l\alpha_j + (1-a)\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \alpha_l\beta_j}{a\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \beta_l + (1-a)\sum_{l\in\mathcal{N}_i^A(t_k^i)\cup\{i\}} \alpha_l}, \quad j \in \mathcal{N}_i^A(t_k^i) \cup \{i\}. \tag{7.1}$$

Clearly, $\gamma_j \in [\alpha_i, \beta_j] \subset [W^L, W^U]$ and thus the above equation implies that $\varsigma \in \mathcal{D}_k^i$. ∎

Proof of Corollary 2.3. This corollary can be proved by a finite-step deduction. We use a simple example to illustrate the deduction process. Suppose that agent j is the only neighbor of agent i. Then for any $\xi \in \mathcal{D}_k^i$, there exist $W^L \leq \alpha, \beta \leq W^U$ such that $\xi = \frac{1}{\alpha+\beta}(\alpha x_i + \beta x_j)$. For α, we can always find some $a \in [0,1]$ such that

$$\alpha = \frac{a(W^L+\beta)W^U + (1-a)(W^U+\beta)W^L}{a(W^L+\beta)+(1-a)(W^U+\beta)}.$$

In addition,

$$\beta = \frac{a(W^L+\beta)\beta + (1-a)(W^U+\beta)\beta}{a(W^L+\beta)+(1-a)(W^U+\beta)}.$$

Recalling (7.1), we have

$$\xi = (1-a)\frac{1}{W^L+\beta}(W^L x_i + \beta x_j) + a\frac{1}{W^U+\beta}(W^U x_i + \beta x_j),$$

which implies that

$$\xi \in \operatorname{co}\left\{\frac{1}{W^L+\beta}(W^L x_i + \beta x_j), \frac{1}{W^U+\beta}(W^U x_i + \beta x_j)\right\}.$$

Again by the same arguments,

$$\begin{aligned}
&\frac{1}{W^L+\beta}(W^L x_i + \beta x_j)\\
&\qquad \in \operatorname{co}\left\{\frac{1}{W^L+W^L}(W^L x_i + W^L x_j), \frac{1}{W^L+W^U}(W^L x_i + W^U x_j)\right\};\\
&\frac{1}{W^U+\beta}(W^U x_i + \beta x_j)\\
&\qquad \in \operatorname{co}\left\{\frac{1}{W^U+W^L}(W^U x_i + W^L x_j), \frac{1}{W^U+W^U}(W^U x_i + W^U x_j)\right\}.
\end{aligned}$$

Clearly, the four vectors in the right two big brackets belong to $\operatorname{co}(\tilde{\mathcal{D}}_k^i)$ and thus $\xi \in \operatorname{co}(\tilde{\mathcal{D}}_k^i)$. The general case can be proved similarly. ◻

Proof of Lemma 3.3. *Proof of* (F1). By the definition of $\mathcal{D}_k^i$,

$$\xi = x_i(t_k^i) + \frac{\sum_{j\in\mathcal{N}_i^A(t_k^i)} W^L\left(x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i)\right)}{|\mathcal{N}_i^A(t_k^i)|W^L + W^U} \in \mathcal{D}_k^i,$$

where $|\mathcal{N}_i^A(t_k^i)|$ denotes the number of elements in set $\mathcal{N}_i^A(t_k^i)$. So by assumption (A6),

$$\begin{aligned}
|x_i(t_{k+1}^i) - x_i(t_k^i)| &\leq |x_i(t_k^i) - \xi|\\
&\leq \frac{\sum_{j\in\mathcal{N}_i^A(t_k^i)} W^L |x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i)|}{|\mathcal{N}_i^A(t_k^i)|W^L + W^U}\\
&\leq \frac{R_{\max}}{1+\frac{W^U}{(n-1)W^L}} \leq S \text{ (by (A5))}.
\end{aligned}$$

Proof of (F2). Without loss of generality, suppose that $x_j(t_k^i - \tau_k^{ij}) \geq x_i(t_k^i) + D$. To prove (F2), by assumptions (A6) and (A7), it suffices to prove that there exists some $\xi \in \mathcal{D}_k^i$ such that $\xi \geq x_i(t_k^i)$. Choose

$$\xi = x_i(t_k^i) + \frac{\sum_{l \in \mathcal{N}_i^A(t_k^i), l \neq j} W^L \left(x_l(t_k^i - \tau_k^{il}) - x_i(t_k^i)\right) + W^U(x_j(t_k^i - \tau_k^{ij}) - x_i(t_k^i))}{|\mathcal{N}_i^A(t_k^i)| W^L + W^U}.$$

So $\xi \in \mathcal{D}_k^i$ and by assumption (A5),

$$\xi - x_i(t_k^i) \geq \frac{-(|\mathcal{N}_i^A(t_k^i)| - 1) W^L R_{\max} + W^U D}{|\mathcal{N}_i^A(t_k^i)| W^L + W^U} \geq 0.$$

Proof of (F3). Fact (F3) is a consequence of assumptions (A1), (A2), and (A7) and fact (F1). ◻

Proof of Theorem 3.4. This theorem will be proved by contradiction.

(AE) Suppose that there exist time $t_e > 0$ (subscript "e" means "escape") and $\varepsilon > 0$ such that $|x_i(t) - x_j(t)| \leq R$ for all $t \leq t_e$ and $|x_i(t) - x_j(t)| > R$ for all $t_e < t \leq t_e + \varepsilon$.

We will show that the above assumption leads to contradictions.

First, we recall the delay operator $\delta(\cdot)$, defined in subsection 4.1. Without loss of generality, assume that $t^j_{\delta(j,t_e)} \leq t^i_{\delta(i,t_e)} \leq t_e$ and $x_i(t_e) < x_j(t_e)$. (The other cases can be proved similarly.) For simplicity, denote $k_i = \delta(i, t_e)$ and $k_j = \delta(j, t_e)$. By fact (F1) and that $R \geq D + (L+2)S$, assumption (AE) implies that

$$x_j(t^j_{k_j}) \geq x_i(t^i_{k_i}) + LS + D. \tag{7.2}$$

Case 1. $t^i_{k_i} = t^j_{k_j} \leq t_e$.

By inequality (7.2) and fact (F3), $|x_i(t^i_{k_i}) - x_j(t^i_{k_i} - \tau^{ij}_{k_i})| \geq D$ and thus by fact (F2), $x_i(t) \geq x_i(t^i_{k_i})$ for any t with $t^i_{k_i} \leq t \leq t^i_{k_i+1}$. Similarly, $x_j(t) \leq x_j(t^j_{k_j})$ for any t with $t^j_{k_j} \leq t \leq t^j_{k_j+1}$. Therefore, $|x_i(t) - x_j(t)| \leq |x_i(t^i_{k_i}) - x_j(t^j_{k_j})| \leq R$ for any t with $t_e < t \leq \min\{t^i_{k_i+1}, t^j_{k_j+1}\}$, which contradicts assumption (AE).

Case 2. $t^j_{k_j} < t^i_{k_i} \leq t_e$.

Case 2.1. $x_j(t^j_{k_j+1}) > x_j(t^j_{k_j})$, namely, agent j moves right after update time $t^j_{k_j}$.

By fact (F2), $x_j(t^j_{k_j}) - x_i(t^j_{k_j} - \tau^{ji}_{k_j}) < D$, and thus by fact (F3),

$$x_i(t^j_{k_j}) > x_j(t^j_{k_j}) - D - LS. \tag{7.3}$$

Let $k = \delta(i, t^j_{k_j})$. Then there exists $k' \geq 1$ such that $t^i_k \leq t^j_{k_j} < t^i_{k+1} < \cdots < t^i_{k+k'} = t^i_{k_i}$. Then by fact (F1) and inequality (7.3),

$$x_i(t^i_{k+1}) > x_j(t^j_{k_j}) - D - (L+1)S. \tag{7.4}$$

On the other hand, by fact (F3) and the assumption that agent j moves right after time $t^j_{k_j}$,

$$\min\{x_j(t^i_{k+1} - \tau^{ij}_{k+1}), x_j(t^i_{k+2} - \tau^{ij}_{k+2}), \ldots, x_j(t^i_{k+k'} - \tau^{ij}_{k+k'})\} \geq x_j(t^j_{k_j}) - LS. \tag{7.5}$$

(1) If all $x_i(t^i_{k+1}), x_i(t^i_{k+2}), \ldots, x_i(t^i_{k+k'+1})$ are not less than $x_j(t^j_{k_j}) - D - (L+1)S$, then by fact (F1), $|x_i(t) - x_j(t)| \leq D + (L+2)S \leq R$ for $t_e < t \leq \min\{t^i_{k_i+1}, t^j_{k_j+1}\}$, which contradicts assumption (AE).

(2) Otherwise, by inequality (7.4), there exists k'', $1 < k'' \leq k'+1$, such that

$$x_i(t^i_{k+k''}) < x_j(t^j_{k_j}) - D - (L+1)S \tag{7.6}$$

and

$$x_i(t^i_{k+k''-1}) \geq x_j(t^j_{k_j}) - D - (L+1)S. \tag{7.7}$$

And by fact (F1), it follows from the inequality (7.6) that

$$x_i(t^i_{k+k''-1}) < x_j(t^j_{k_j}) - D - LS.$$

However, the above inequality and inequality (7.5) imply that

$$x_j(t^i_{k+k''-1} - \tau^{ij}_{k+k''-1}) - x_i(t^i_{k+k''-1}) > D.$$

By fact (F2) and inequality (7.7),

$$x_i(t^i_{k+k''}) \geq x_i(t^i_{k+k''-1}) \geq x_j(t^j_{k_j}) - D - (L+1)S,$$

which contradicts inequality (7.6).

Case 2.2. $x_j(t^j_{k_j+1}) \leq x_j(t^j_{k_j})$, namely, agent j keep stationary or moves left after update time $t^j_{k_j}$.

By fact (F1) and the assumption that $t^i_{k_i} > t^j_{k_j}$, $|x_j(t^j_{k_j}) - x_j(t^i_{k_i})| \leq S$ and thus by fact (F3),

$$|x_j(t^i_{k_i} - \tau^{ij}_{k_i}) - x_j(t^j_{k_j})| \leq (L+1)S. \tag{7.8}$$

(1) If $x_i(t^i_{k_i+1}) < x_i(t^i_{k_i})$, then by fact (F2), $x_j(t^i_{k_i} - \tau^{ij}_{k_i}) < x_i(t^i_{k_i}) + D$. By inequalities (7.8) and (7.2),

$$x_j(t^j_{k_j}) - (LS+D) \geq x_i(t^i_{k_i}) > x_j(t^j_{k_j}) - ((L+1)S + D).$$

Therefore,

$$x_j(t^j_{k_j}) - (LS+D) > x_i(t^i_{k_i+1}) > x_j(t^j_{k_j}) - ((L+2)S + D),$$

which yields that for any t with $t^i_{k_i} \leq t \leq t^i_{k_i+1}$,

$$LS + D \leq x_j(t^j_{k_j}) - x_i(t) < (L+2)S + D \leq R.$$

So $(v_i, v_j) \in \mathcal{G}(t)$ when $t_e < t \leq \min\{t^i_{k_i+1}, t^j_{k_j+1}\}$, which contradicts assumption (AE).

(2) If $x_i(t^i_{k_i+1}) \geq x_i(t^i_{k_i})$, then by assumption (A7), for any t with $t_e < t \leq \min\{t^i_{k_i+1}, t^j_{k_j+1}\}$, $|x_i(t) - x_j(t)| \leq |x_i(t_e) - x_j(t_e)| = R$, contradicting assumption (AE). ☐

Proof of Corollary 3.5. The corollary can also be proved by contradiction. Suppose that there exist time $t_e > 0$ and $\varepsilon > 0$ such that $(v_i, v_j) \in \mathcal{G}(t)$ for all $t \leq t_e$ and $(v_i, v_j) \notin \mathcal{G}(t)$ for all $t_e < t \leq t_e + \varepsilon$. This assumption implies that there exist a coordinate direction, for example, x-axis, and $\varepsilon' > 0$ such that $|x'_i(t) - x'_j(t)| \leq \frac{\sqrt{2}R}{2}$ for any $t \leq t_e$ and $|x'_i(t) - x'_j(t)| > \frac{\sqrt{2}R}{2}$ for any $t_e < t \leq t_e + \varepsilon'$, where x'_i and x'_j represent the coordinates of agents i and j with respect to the x-axis.

Since the dynamics in the directions of the x-axis and y-axis are decoupled, with the same arguments as in proving Theorem 3.4, the above assumption results in contradictions, and thus the corollary holds. ☐

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