Physics Letters A 373 (2009) 342–348

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Improved stability criteria for neural networks with time-varying delay [☆]

Jian Sun [a,b,*], G.P. Liu [a,c], Jie Chen [b], D. Rees [a]

[a] Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, UK
[b] Department of Automatic Control, School of Information Science and Technology, Beijing Institute of Technology, Beijing 100081, China
[c] Key Laboratory of Complex System and Intelligent Science, Institute of Automation, Chinese Academy of Sciences, Beijing 100190, China

ARTICLE INFO

Article history:
Received 9 July 2008
Received in revised form 7 November 2008
Accepted 20 November 2008
Available online 30 November 2008
Communicated by R. Wu

PACS:
07.05.Mh
02.30.Ks

Keywords:
Delay-dependent stability
Time-varying delay
Neural networks
Linear matrix inequality (LMI)

ABSTRACT

The problem of the stability analysis of neural networks with time-varying delay is considered in this Letter. By constructing a new augmented Lyapunov functional which contains a triple-integral term, an improved delay-dependent stability criterion is derived in terms of LMI using the free-weighting matrices method. The rate-range of the delay is also considered in the derivation of the criterion. Numerical examples are presented to illustrate the effectiveness of the proposed method.

1. Introduction

Recently, stability of the delayed neural network has been extensively studied. The existing stability criteria can be classified into two categories, namely, delay-independent ones [1–5] and delay-dependent ones [6–17]. Since delay-independent ones are usually more conservative than delay-dependent ones especially when the delay is small, delay-dependent stability criteria for delayed neural networks have received much attention.

In [6], the descriptor system approach was applied to derive the delay-dependent exponential stability conditions for delayed neural networks. An improved stability criterion was proposed in [7] by constructing a new Lyapunov functional and using the S-procedure. Using the free-weighting matrices method, a new delay-dependent stability criterion for neural networks with time-varying delay is derived in [8]. In the above papers, some useful terms were ignored when estimating the upper bound of the derivative of the Lyapunov functional. So some less conservative stability criteria were proposed in [9] by considering those useful terms and using the free-weighting matrices method. The stability of neural networks with time-varying interval delay were considered in [18] where the relationship between the time-varying delay and its lower and upper bound was taken into account and an elegant result was derived. However, there still exists room for further improvements.

It can be seen that the Lyapunov functional introduced in [18] only contains some integral terms, for example $\int_{t-h}^{t} z^T(s)Qz(s)\,ds$, and double-integral terms, for example $\int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z\dot{z}(s)\,ds\,d\theta$. If a triple-integral term is introduced in the Lyapunov functional, what results can be obtained? This idea motivates this study. In addition, the lower bound of the delay-derivative was not considered in the above publications. If information on the lower bound of the delay-derivative is used in the derivation of stability criterion, a less conservative result may be obtained.

☆ This work is supported by National Science Foundation of China under Grant 60528002 and by the Beijing Education Committee Cooperation Building Foundation Project.
* Corresponding author at: Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, UK. Tel.: +44 1443 482817; fax: +44 1443 482514.
E-mail addresses: jsun@glam.ac.uk (J. Sun), gpliu@glam.ac.uk (G.P. Liu), chenjie@bit.edu.cn (J. Chen), drees@glam.ac.uk (D. Rees).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.11.048

2. Problem formulation

Consider the following delayed neural network:

$$\dot{x}(t) = -Cx(t) + Ag\big(x(t)\big) + A_1 g\big(x\big(t-\tau(t)\big)\big) + u \tag{1}$$

where $x(\cdot) = [x_1(\cdot)\ x_2(\cdot)\ \cdots\ x_n(\cdot)]^T$ is the neuron state vector, $g(x(\cdot)) = [g_1(x_1(\cdot))\ g_2(x_2(\cdot))\ \cdots\ g_n(x_n(\cdot))]^T$ is the neuron activation function, and $u = [u_1\ u_2\ \cdots\ u_n]^T$ is a constant input vector. $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ with $c_i > 0$, $i = 1, 2, \ldots, n$, is a diagonal matrix representing self-feedback term, A is the connection weight matrix and A_1 is the delayed connection weight matrix. The delay $\tau(t)$ is a time-varying differentiable function satisfying

$$0 \leqslant \tau(t) \leqslant h \tag{2}$$

and

$$\mu_1 \leqslant \dot{\tau}(t) \leqslant \mu_2 < 1 \tag{3}$$

where $h \geqslant 0$, μ_1 and μ_2 are constants. It is assumed that each neuron activation function, $g_i(\cdot)$, $i = 1, 2, \ldots, n$, is nondecreasing, bounded and satisfying the following condition:

$$0 \leqslant \frac{g_i(x) - g_i(y)}{x - y} \leqslant m_i \quad \forall x, y \in \mathbb{R},\ x \neq y,\ i = 1, 2, \ldots, n, \tag{4}$$

where m_i, $i = 1, 2, \ldots, n$, are positive constants.

Assuming that $x^* = [x_1^*\ x_2^*\ \cdots\ x_n^*]$ is the equilibrium point of (1) whose uniqueness has been given in [17] and using the transformation $z(\cdot) = x(\cdot) - x^*$, (1) can be converted to the following error system:

$$\dot{z}(t) = -Cz(t) + Af\big(z(t)\big) + A_1 f\big(z\big(t-\tau(t)\big)\big) \tag{5}$$

where $z(\cdot) = [z_1(\cdot)\ z_2(\cdot)\ \cdots\ z_n(\cdot)]^T$ is the state vector, $f(z(\cdot)) = [f_1(z_1(\cdot))\ f_2(z_2(\cdot))\ \cdots\ f_n(z_n(\cdot))]^T$, and $f_i(z_i(\cdot)) = g_i(z_i(\cdot) + x_i^*) - g_i(x_i^*)$, $i = 1, 2, \ldots, n$. According to (4), one can obtain that the functions $f_i(\cdot)$, $i = 1, 2, \ldots, n$, satisfy the following condition:

$$0 \leqslant \frac{f_i(z_i)}{z_i} \leqslant m_i, \quad f_i(0) = 0,\ \forall z_i \neq 0,\ i = 1, 2, \ldots, n, \tag{6}$$

which is equivalent to

$$f_i(z_i)\big[f_i(z_i) - m_i z_i\big] \leqslant 0, \quad f_i(0) = 0,\ i = 1, 2, \ldots, n. \tag{7}$$

3. Main results

In this section, a new augmented Lyapunov functional is constructed and a less conservative delay-dependent stability criterion is obtained.

Theorem 1. *For given scalars $h \geqslant 0$, μ_1 and μ_2, system (5) is asymptotically stable for any time-varying delay satisfying (2) and (3) if there exist matrices $R_l = R_l^T > 0$, $S_l = S_l^T > 0$, $l = 1, 2$, $U = U^T > 0$, $P = P^T = [P_{ij}]_{4\times4} > 0$, $Q = Q^T = [Q_{ij}]_{3\times3} > 0$, $X = X^T = [X_{ij}]_{7\times7} \geqslant 0$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geqslant 0$, $W_k = \mathrm{diag}\{W_{1k}, W_{2k}, \ldots, W_{nk}\} \geqslant 0$, $k = 1, 2$, and any matrices L, N, and H with appropriate dimensions such that the following LMIs holds*:

$$\begin{bmatrix} \Xi & A_c^T Y & \frac{1}{2}h^2 H \\ Y A_c & -Y & 0 \\ \frac{1}{2}h^2 H^T & 0 & -\frac{1}{2}h^2 U \end{bmatrix} < 0, \tag{8}$$

$$\Pi_1 = \begin{bmatrix} X & \Gamma + H & L \\ \Gamma^T + H^T & S_1 & 0 \\ L^T & 0 & S_2 \end{bmatrix} \geqslant 0, \tag{9}$$

$$\Pi_2 = \begin{bmatrix} X & \Gamma + H & N \\ \Gamma^T + H^T & S_1 & 0 \\ N^T & 0 & S_2 \end{bmatrix} \geqslant 0 \tag{10}$$

where

$$\Xi = [\Xi_{ij}]_{7\times7},$$

$$\Xi_{11} = -P_{11}C - C^T P_{11} - Q_{12}C - C^T Q_{12}^T + P_{14} + P_{14}^T + Q_{11} + R_1 + hS_1 + L_1 + L_1^T + hH_1 + hH_1^T + hX_{11},$$

$$\Xi_{12} = -L_1 + L_2^T + N_1 - C^T P_{12} + P_{24}^T + hH_2^T + hX_{12},$$

$$\Xi_{13} = P_{12} + L_3^T + hH_3^T + hX_{13},$$

$$\Xi_{14} = -C^T P_{13} + P_{34}^T + L_4^T - P_{14} - N_1 + hH_4^T + hX_{14},$$

$$\Xi_{15} = P_{13} + L_5^T + hH_5^T + hX_{15},$$

$$\Xi_{16} = P_{11}A + Q_{12}A + Q_{13} - C^T Q_{23} + L_6^T + hH_6^T - C^T \Lambda + MW_1 + hX_{16},$$

$$\Xi_{17} = P_{11}A_1 + Q_{12}A_1 + L_7^T + hH_7^T + hX_{17},$$
$$\Xi_{22} = -(1-\mu_2)Q_{11} - L_2 - L_2^T + N_2 + N_2^T + hX_{22},$$
$$\Xi_{23} = P_{22} - Q_{12} - L_3^T + N_3^T + hX_{23},$$
$$\Xi_{24} = -L4^T + N_4^T - P_{24} - N_2 + hX_{24},$$
$$\Xi_{25} = -L_5^T + N_5^T + P_{23} + hX_{25},$$
$$\Xi_{26} = -L_6^T + N_6^T + P_{12}^T A + hX_{26},$$
$$\Xi_{27} = -L_7^T + N_7^T + P_{12}^T A_1 - (1-\mu_2)Q_{13} + MW_2 + hX_{27},$$
$$\Xi_{33} = -Q_{22}/(1-\mu_1) + hX_{33},$$
$$\Xi_{34} = P_{23} - N_3 + hX_{34},$$
$$\Xi_{35} = hX_{35},$$
$$\Xi_{36} = hX_{36},$$
$$\Xi_{37} = -Q_{23} + hX_{37},$$
$$\Xi_{44} = -P_{34} - P_{34}^T - R_1 - N_4 - N_4^T + hX_{44},$$
$$\Xi_{45} = P_{33} - N_5^T + hX_{45},$$
$$\Xi_{46} = -N_6^T + P_{13}^T A + hX_{46},$$
$$\Xi_{47} = -N_7^T + P_{13}^T A_1 + hX_{47},$$
$$\Xi_{55} = -R_2 + hX_{55},$$
$$\Xi_{56} = hX_{56},$$
$$\Xi_{57} = hX_{57},$$
$$\Xi_{66} = \Lambda A + A^T\Lambda + A^T Q_{23} + Q_{23}^T A + Q_{33} - 2W_1 + hX_{66},$$
$$\Xi_{67} = \Lambda A_1 + Q_{23}^T A_1 + hX_{67},$$
$$\Xi_{77} = -(1-\mu_2)Q_{33} - 2W_2 + hX_{77},$$
$$Y = Q_{22} + R_2 + hS_2 + \frac{h^2}{2}U,$$
$$A_c = [\,-C \quad 0 \quad 0 \quad 0 \quad A \quad A_1\,],$$
$$M = \mathrm{diag}\{m_1, m_2, \ldots, m_n\},$$
$$\Gamma^T = \begin{bmatrix} P_{14}^T C - P_{44} & 0 & -P_{24}^T & P_{44} & -P_{34}^T & -P_{14}^T A & -P_{14}^T A_1 \end{bmatrix}.$$

Proof. Construct the following Lyapunov functional

$$V\big(z(t)\big) = V_1\big(z(t)\big) + V_2\big(z(t)\big) + V_3\big(z(t)\big) + V_4\big(z(t)\big) + V_5\big(z(t)\big) \tag{11}$$

with

$$V_1\big(z(t)\big) = \zeta^T(t)P\zeta(t) + 2\sum_{i=1}^{n}\lambda_i \int_0^{z_i} f_i(s)\,ds,$$
$$V_2\big(z(t)\big) = \int_{t-\tau(t)}^{t} \xi^T(s)Q\xi(s)\,ds,$$
$$V_3\big(z(t)\big) = \int_{t-h}^{t} z^T(s)R_1 z(s)\,ds + \int_{t-h}^{t} \dot{z}^T(s)R_2\dot{z}(s)\,ds,$$
$$V_4\big(z(t)\big) = \int_{-h}^{0}\int_{t+\theta}^{t} z^T(s)S_1 z(s)\,ds\,d\theta + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)S_2\dot{z}(s)\,ds\,d\theta,$$
$$V_5\big(z(t)\big) = \int_{-h}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{z}^T(s)U\dot{z}(s)\,ds\,d\lambda\,d\theta$$

where $\zeta(t) = \mathrm{col}\{z(t), z(t-\tau(t)), z(t-h), \int_{t-h}^{t} z(s)\,ds\}$, $\xi(s) = \mathrm{col}\{z(s), \dot{z}(s), f(z(s))\}$. Taking the derivative of $V(z(t))$ along the trajectories of system (5) yields

$$\dot{V}_1\big(z(t)\big) = 2\zeta^T(t)P\dot{\zeta}(t) + 2\sum_{i=1}^{n}\lambda_i f_i\big(z_i(t)\big)\dot{z}_i(t) = 2\zeta^T(t)P\dot{\zeta}(t) + 2f^T\big(z(t)\big)\Lambda\dot{z}(t), \tag{12}$$

$$\dot{V}_2\big(z(t)\big) = \xi^T(t)Q\xi(t) - \big(1-\dot{\tau}(t)\big)\xi^T\big(t-\tau(t)\big)Q\xi\big(t-\tau(t)\big), \tag{13}$$

$$\dot{V}_3\big(z(t)\big) = z^T(t)R_1z(t) - z^T(t-h)R_1z(t-h) + \dot{z}^T(t)R_2\dot{z}(t) - \dot{z}^T(t-h)R_2\dot{z}(t-h), \tag{14}$$

$$\begin{aligned}\dot{V}_4\big(z(t)\big) &= hz^T(t)S_1z(t) - \int_{t-h}^{t} z^T(s)S_1z(s)\,ds + h\dot{z}^T(t)S_2\dot{z}(t) - \int_{t-h}^{t}\dot{z}^T(s)S_2\dot{z}(s)\,ds \\ &= hz^T(t)S_1z(t) - \int_{t-\tau(t)}^{t} z^T(s)S_1z(s)\,ds - \int_{t-h}^{t-\tau(t)} z^T(s)S_1z(s)\,ds + h\dot{z}^T(t)S_2\dot{z}(t) - \int_{t-\tau(t)}^{t}\dot{z}^T(s)S_2\dot{z}(s)\,ds - \int_{t-h}^{t-\tau(t)}\dot{z}^T(s)S_2\dot{z}(s)\,ds,\end{aligned} \tag{15}$$

$$\dot{V}_5\big(z(t)\big) = \frac{1}{2}h^2\dot{z}^T(t)U\dot{z}(t) - \int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}^T(s)U\dot{z}(s)\,ds\,d\theta. \tag{16}$$

In addition, from (7) one can obtained that

$$f_i\big(z_i(t)\big)\big[f_i\big(z_i(t)\big) - m_iz_i(t)\big] \leqslant 0, \quad i = 1,2,\ldots,n, \tag{17}$$

$$f_i\big(z_i\big(t-\tau(t)\big)\big)\big[f_i\big(z_i\big(t-\tau(t)\big)\big) - m_iz_i\big(t-\tau(t)\big)\big] \leqslant 0, \quad i = 1,2,\ldots,n. \tag{18}$$

It is clear that the following inequality holds for any $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \ldots, W_{ni}\} \geqslant 0$, $i = 1,2$.

$$\begin{aligned}0 &\leqslant -2\sum_{i=1}^{n}W_{i1}f_i\big(z_i(t)\big)\big[f_i\big(z_i(t)\big) - m_iz_i(t)\big] - 2\sum_{i=1}^{n}W_{i2}f_i\big(z_i\big(t-\tau(t)\big)\big)\big[f_i\big(z_i\big(t-\tau(t)\big)\big) - m_iz_i\big(t-\tau(t)\big)\big] \\ &= 2z^T(t)MW_1f\big(z(t)\big) - 2f^T\big(z(t)\big)W_1f\big(z(t)\big) + 2z^T\big(t-\tau(t)\big)MW_2f\big(z\big(t-\tau(t)\big)\big) - 2f^T\big(z\big(t-\tau(t)\big)\big)W_2f\big(z\big(t-\tau(t)\big)\big).\end{aligned} \tag{19}$$

Similar to [9], the following equations hold:

$$\Delta_1 := 2\theta^T(t)L\left[z(t) - z\big(t-\tau(t)\big) - \int_{t-\tau(t)}^{t}\dot{z}(s)\,ds\right] = 0, \tag{20}$$

$$\Delta_2 := 2\theta^T(t)N\left[z\big(t-\tau(t)\big) - z(t-h) - \int_{t-h}^{t-\tau(t)}\dot{z}(s)\,ds\right] = 0, \tag{21}$$

$$\Delta_3 := 2\theta^T(t)H\left[hz(t) - \int_{t-\tau(t)}^{t}z(s)\,ds - \int_{t-h}^{t-\tau(t)}z(s)\,ds - \int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}(s)\,ds\,d\theta\right] = 0, \tag{22}$$

$$\Delta_4 = h\theta^T(t)X\theta(t) - \int_{t-\tau(t)}^{t}\theta^T(t)X\theta(t)\,ds - \int_{t-h}^{t-\tau(t)}\theta^T(t)X\theta(t)\,ds = 0 \tag{23}$$

where $\theta(t) = \mathrm{col}\{z(t), z(t-\tau(t)), \dot{z}(t-\tau(t))(1-\dot{\tau}(t)), z(t-h), \dot{z}(t-h), f(z(t)), f(z(t-\tau(t)))\}$ and

$$-2\theta^T(t)H\int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}(s)\,ds\,d\theta \leqslant \frac{1}{2}h^2\theta^T(t)HU^{-1}H^T\theta(t) + \int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}^T(s)U\dot{z}(s)\,ds\,d\theta. \tag{24}$$

From (12)–(16) and (20)–(23), it is easy to obtain that

$$\dot{V}\big(z(t)\big) = \sum_{i=1}^{5}\dot{V}_i\big(z(t)\big) + \sum_{j=1}^{4}\Delta_j. \tag{25}$$

Adding both sides of (19) into both sides of (25) and using (24) yield

$$\dot{V}\big(z(t)\big) \leqslant \theta^T(t)\left[\hat{\Xi} + A_c^TYA_c + \frac{1}{2}h^2HU^{-1}H^T\right]\theta(t) - \int_{t-\tau(t)}^{t}\theta^T(t,s)\Pi_1\theta(t,s)\,ds - \int_{t-h}^{t-\tau(t)}\theta^T(t,s)\Pi_2\theta(t,s)\,ds \tag{26}$$

where $\theta^T(t,s) = [\theta^T(t)\ z^T(s)\ \dot{z}^T(s)]$, $\hat{\Xi} = [\hat{\Xi}_{ij}]_{7\times7}$ with $\hat{\Xi}_{22} = -(1-\dot{\tau}(t))Q_{11} - M_2 - M_2^T + N_2 + N_2^T + hX_{22}$, $\hat{\Xi}_{27} = -L_7^T + N_7^T + P_{12}^TA_1 - (1-\dot{\tau}(t))Q_{13} + MW_2 + hX_{27}$, $\hat{\Xi}_{33} = Q_{22}/(1-\tau(t)) + hX_{33}$, $\hat{\Xi}_{77} = -(1-\dot{\tau}(t))Q_{33} - 2W_2 + hX_{77}$, and the others $\hat{\Xi}_{ij}$ are the same as Ξ_{ij}.

From (3), it is easy to see that $\hat{\Xi} \leqslant \Xi$. So, if $\Xi + A_c^TYA_c + \frac{1}{2}h^2HU^{-1}H^T < 0$, $\Pi_1 \geqslant 0$ and $\Pi_2 \geqslant 0$, $\dot{V}(z(t)) < -\varepsilon\|z(t)\|^2$ for a sufficiently small $\varepsilon > 0$ such that system (5) is asymptotically stable. By Schur complements, $\Xi + A_c^TYA_c + \frac{1}{2}h^2HU^{-1}H^T < 0$ is equivalent to (8). The proof is completed. □

Remark 2. Compared with the existing augmented Lyapunov functional [19], the proposed one contains a triple-integral term. It can be seen that the augmented vector $\zeta(t)$ in the proposed Lyapunov functional (11) contains an integral term $\int_{t-h}^{t} x(s)\,ds$. Both this term and the triple-integral term play important roles in the reduction in conservativeness. Through some numerical examples, it can be found that if $\int_{t-h}^{t} x(s)\,ds$ is not introduced in the augmented vector, the introduction of the triple-integral term does not contribute to a further reduction in conservativeness. On the other hand, if only the integral term $\int_{t-h}^{t} x(s)\,ds$ is introduced in the augmented vector with the triple-integral term omitted in the Lyapunov functional (11), then this Lyapunov functional does not lead to a less conservative result.

Remark 3. It is easy to see that the derivative of $\zeta^T(t)P\zeta(t)$ has some terms containing $1-\dot{\tau}(t)$. In order to estimate it, a bounding technique is used in [19] (see Eq. (22)), which introduces some conservativeness. However, the method proposed in this Letter is much different. In the definition of $\theta(t)$, it is $\dot{z}^T(t-\tau(t))(1-\dot{\tau}(t))$ but not $\dot{z}^T(t-\tau(t))$ that is introduced. And this definition of $\theta(t)$ can absorb some $1-\dot{\tau}(t)$, so $\hat{\Xi}$ contains less $1-\dot{\tau}(t)$, which makes the estimation of the upper bound of the derivative of the Lyapunov functional much easier.

Remark 4. Clearly, the stability condition in Theorem 1 is delay-dependent. Furthermore, it is also dependent on both the upper bound and the lower bound of the delay-derivative. And, there is no additional restriction on μ_1. However, the delay-derivative in [19] is restrict to $|\dot{\tau}(t)| \leqslant \mu$ and $\mu < 1$.

Before finishing this section, it should be pointed out that the proposed method can also be used to develop a delay-independent stability criterion. If letting $S_1 = \varepsilon_1 I$, $S_2 = \varepsilon_2 I$, $U = \varepsilon_3 I$ and $P_{44} = \varepsilon_4 I$, where ε_i, $i = 1,\dots,4$, are sufficiently small positive scalars, and setting $P_{j4} = 0$, $j = 1,\dots,3$, in the Lyapunov functional (11), the following delay-independent result is obtained.

Corollary 5. *For given scalars μ_1 and μ_2 and any $h \geqslant 0$, system (5) is asymptotically stable for any time-varying delay satisfying (2) and (3) if there exist matrices $P = P^T = [P_{ij}]_{3\times3} > 0$, $R_1 > 0$, $R_2 > 0$ and $Q = Q^T = [Q_{ij}]_{3\times3} > 0$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} \geqslant 0$, and $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \dots, W_{ni}\} \geqslant 0$, $i = 1, 2$, such that the following LMI holds*

$$\begin{bmatrix} \Sigma_{11} & -C^T P_{12} & P_{12} & -C^T P_{13} & P_{13} & \Sigma_{16} & \Sigma_{17} & -C^T Y \\ * & -(1-\mu_2)Q_{11} & P_{22}-Q_{12} & 0 & P_{23} & P_{12}^T A & \Sigma_{27} & 0 \\ * & * & -\frac{Q_{22}}{1-\mu_1} & P_{23} & 0 & 0 & -Q_{23} & 0 \\ * & * & * & -R_1 & P_{33} & P_{13}^T A & P_{13}^T A_1 & 0 \\ * & * & * & * & -R_2 & 0 & 0 & 0 \\ * & * & * & * & * & \Sigma_{66} & \Sigma_{67} & A^T Y \\ * & * & * & * & * & * & \Sigma_{77} & A_1^T Y \\ * & * & * & * & * & * & * & -Y \end{bmatrix} < 0 \tag{27}$$

where

$$\Sigma_{11} = -P_{11}C - C^T P_{11} + Q_{11} - Q_{12}C - C^T Q_{12}^T,$$

$$\Sigma_{16} = P_{11}A + Q_{12}A + Q_{13} - C^T Q_{23} - C^T \Lambda + MW_1,$$

$$\Sigma_{17} = P_{11}A_1 + Q_{12}A_1,$$

$$\Sigma_{27} = P_{12}^T A_1 - (1-\mu_2)Q_{13} + MW_2,$$

$$\Sigma_{66} = \Lambda A + A^T \Lambda + A^T Q_{23} + Q_{23}^T A + Q_{33} - 2W_1,$$

$$\Sigma_{67} = \Lambda A_1 + Q_{23}^T A_1,$$

$$\Sigma_{77} = -(1-\mu_2)Q_{33} - 2W_2,$$

$$Y = Q_{22} + R_2,$$

$$M = \mathrm{diag}\{m_1, m_2, \dots, m_n\}.$$

Remark 6. Clearly, Corollary 5 is delay-independent. However, it is dependent on the rate-range of the delay, that is, it is dependent on both the upper bound and the lower bound of the delay-derivative.

4. Numerical examples

In this section, two numerical examples are presented to show the less conservativeness of the proposed methods.

Example 1. Consider the following delayed neural network with [17]

$$C = \mathrm{diag}\{1.2769, 0.6231, 0.9230, 0.4480\}, \qquad A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9634 & -0.5015 \end{bmatrix},$$

Table 1
Upper bounds of h for different μ.

μ	0.1	0.5	0.9
Hua et al. [10]	3.2775	2.1502	1.3164
He et al. [9]	3.2793	2.2245	1.5847
He et al. [18]	3.3039	2.5376	2.0853
Theorem 1	3.7008	3.1245	2.5979

Table 2
Upper bounds of h for different μ.

μ	0.4	0.45	0.5	0.55
He et al. [18]	3.9972	3.2760	3.0594	2.9814
Case 1	4.2093	3.4515	3.2307	3.1668
Case 2	4.2093	3.4515	3.2307	3.1668
Case 3	4.2093	3.4515	3.2307	3.1668
Theorem 1	4.3814	3.6008	3.3377	3.2350

$$A_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$
$$m_1 = 0.1137, \qquad m_2 = 0.1279, \qquad m_3 = 0.7994, \qquad m_4 = 0.2368.$$

It is assumed that $|\dot{\tau}(t)| \leqslant \mu$. The corresponding upper bounds of h for various μ calculated by Theorem 1 are listed in Table 1 compared with those in [9,10,18]. It can be seen that our results are least conservative. This example has illustrated that the new method proposed in this Letter can lead to less conservative results.

Example 2. Consider the following delayed neural network with

$$C = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$
$$m_1 = 0.3, \qquad m_2 = 0.8.$$

It is assumed that $|\dot{\tau}(t)| \leqslant \mu$. This example is presented to illustrate the statements in Remark 2. If $\int_{t-h}^{t} x(s)\,ds$ is not introduced in the augmented vector $\zeta(t)$ but only the triple-integral term, $\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{z}^T(s)U\dot{z}(s)\,ds\,d\lambda\,d\theta$, is introduced in the Lyapunov functional, a result can be obtained form Theorem 1 directly by setting $P_{i4} = 0$, $i = 1, 2, 3, 4$, and is referred to as Case 1. Similarly, if the triple-integral term is not introduced in the Lyapunov functional but only $\int_{t-h}^{t} x(s)\,ds$ is introduced in the augmented vector, another result can be obtained by setting $H = 0$ and $U = \varepsilon I$ with $\varepsilon > 0$ being a sufficiently small scalar and is referred to as Case 2. If the $\int_{t-h}^{t} x(s)\,ds$ and the triple-integral term are all removed from the Lyapunov functional, another result can be obtained. We call this result Case 3. These three results are all special cases of Theorem 1. Using these three results and Theorem 1, the upper bounds of h for various μ are listed in Table 2 compared with results in [18].

From Table 2, it can be seen that Cases 1, 2 and 3 yield the same results but more conservative than those obtained by Theorem 1. This fact illustrates the statements in Remark 2, that is, the $\int_{t-h}^{t} x(s)\,ds$ and the triple-integral terms should co-exist in the Lyapunov functional. Without either, the other one may not contribute to the further reduction of the conservativeness. Furthermore, it can seen that results obtained by Cases 1, 2 and 3 are still less conservative than those in [18]. This is mainly because the information on the lower bound of the delay-derivative is used in our results.

5. Conclusion

In this Letter, the stability problem of neural networks with time-varying delay has been investigated. A new augmented Lyapunov functional has been introduced and a new method of estimating the upper bound of the derivative of the Lyapunov functional has also been proposed. New stability criteria have been developed. Numerical examples have illustrated the effectiveness of the proposed method.

It should be noted that results in this Letter may involve more computational complexity especially when the dimension of the neural network is large. However, the proposed results may be easily checked due to the availability of high speed processor. How to reduce the number of decision variables may be an important issue for further study.

References

[1] S. Arik, IEEE Trans. Neural Netw. 13 (5) (2002) 1239.
[2] S. Arik, Neural Netw. 17 (2004) 1027.
[3] J.D. Cao, L. Wang, IEEE Trans. Neural Netw. 13 (2002) 457.
[4] J.D. Cao, J. Wang, IEEE Trans. Circuits Syst. I, Reg. Papers 52 (5) (2005) 920.
[5] V. Singh, IEEE Trans. Neural Netw. 15 (1) (2004) 223.
[6] W.-H. Chen, X.-M. Lu, Z.-H. Guan, W.X. Zheng, IEEE Trans. Circuits Syst. II, Exp. Briefs 53 (9) (2006) 837.
[7] Y. He, M. Wu, J.-H. She, IEEE Trans. Neural Netw. 17 (1) (2006) 250.
[8] Y. He, M. Wu, J.-H. She, IEEE Trans. Circuits Syst. II, Exp. Briefs 53 (7) (2006) 553.
[9] Y. He, G.P. Liu, D. Rees, IEEE Trans. Neural Netw. 18 (1) (2007) 310.
[10] C.C. Hua, C.N. Long, X.P. Guan, Phys. Lett. A 352 (2006) 335.

[11] H. Li, B. Chen, Q. Zhou, S. Fang, Phys. Lett. A 372 (2008) 3385.
[12] C.H. Lien, L.Y. Chung, Chaos Solitons Fractals 34 (2007) 1213.
[13] Y.R. Liu, Z. Wang, X.H. Liu, Neurocomputing 71 (2008) 823.
[14] N. Ozcan, S. Arik, IEEE Trans. Circuits Syst. I, Reg. Papers 53 (1) (2006) 166.
[15] J.H. Park, H.J. Cho, Chaos Solitons Fractals 33 (2007) 436.
[16] Z. Wang, Y. Liu, X. Liu, Phys. Lett. A 345 (2005) 299.
[17] S. Xu, J. Lam, D.W.C. Ho, Y. Zou, IEEE Trans. Syst. Man Cybern. Part B 35 (6) (2005) 1317.
[18] Y. He, G.P. Liu, D. Rees, IEEE Trans. Neural Netw. 18 (6) (2007) 1850.
[19] Y. He, Q.-G. Wang, L. Xie, C. Lin, IEEE Trans. Automat. Control 52 (2) (2007) 293.