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Convergence speed of consensus problems over undirected scale-free networks*

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Scale-free networks and consensus behaviour among multiple agents have both attracted much attention. To investigate the consensus speed over scale-free networks is the major topic of the present work. A novel method is developed to construct scale-free networks due to their remarkable power-law degree distributions, while preserving the diversity of network topologies. The time cost or iterations for networks to reach a certain level of consensus is discussed, considering the influence from power-law parameters. They are both demonstrated to be reversed power-law functions of the algebraic connectivity, which is viewed as a measurement on convergence speed of the consensus behaviour. The attempts of tuning power-law parameters may speed up the consensus procedure, but it could also make the network less robust over time delay at the same time. Large scale of simulations are supportive to the conclusions.

Keywords: scale-free networks, consensus, power-law distribution

PACC: 0590, 0250

1. Introduction

The communication topology of the multi-agent system (MAS) has been an important topic for the decade. Building an interaction network whose topology is a complete graph is luxurious and generally wasteful. In search of a robust and cost-effective solution, researchers come up to the basic preferential attachment rules. Then the scale-free networks are re-discovered by the end of last century.[1] A network is “scale-free” if its degree distribution follows a power law, or at least asymptotically. It is reported to be one of the most popular models in multi-agent systems.[1,2] Scale-free networks are regarded to be more robust and immune to the random mutation and perturbation. In Ref. [3], the authors introduced a structural method which could help to distinguish scale-free networks. This provides a mathematical method to measure whether a network is “scale-free”. Many researchers model[1−13] and study the properties[14−23] of scale-free networks. These models introduce new parameters besides the power-law distribution, or lead to quite limited power-law exponent. Furthermore, these models cannot predict the parameters of a network with limited N nodes before it is constructed. Researchers have studied typical scale-free behaviours such as computer virus,[19] epidemic spreading models[20] and opinion spreading dynamics.[21] One of the conclusions is that the peculiar topological features of scale-free network and the absence of small-world properties may determine epidemic spreading speed. Cascading in scale-free networks[22,23] and the tolerance[24] against it is discussed, and protection scheme is proposed in Ref. [25]. The robustness of weighted networks against cascading failure is discussed in Ref. [26]. An optimal weighting scheme to suppress cascades and traffic congestion is stated in Ref. [27]. Literatures discussed the approximate eigenvalues of adjacency matrix and graph Laplacian of scale-free networks.[28−30] The spectral properties are reported to be related the consensus behaviour of a network.[31,32]

Consensus or synchronization is a major technique of MAS applications. Compared with the strategy based on agents’ reactive behaviours, consensus algorithms lead to simpler communication mechanism and less information types—merely the “consensus information”. Many researchers have proposed consensus algorithms. A typical continuous consensus model was presented in Ref. [32], where the concept of solvability of consensus problems was first proposed. In Ref. [33], the authors studied asynchronous consen-

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sus problems of continuous-time agent model with discontinuous information transmission. The authors in Ref. [34] investigated the consensus problem specified for scale-free networks. There are also literatures concerning with nonlinear or chaotic systems.[35−37] Cao *et al.* first introduced the concept of fractional-order consensus algorithms,[38] where a comparison between integer-order and fractional-order consensus algorithms were drawn. One application of network consensus is to solve the diffusion control problem—where and how much neutralizer the mobile actuators should spray.[39] Experimental implements of consensus under directed, possibly switching interaction topologies with a real multi-robot system is given in Ref. [40]. A review of consensus algorithms can be found in Ref. [41]. About synchronization and complex networks, readers can refer to Ref. [42].

Many consensus algorithms have been introduced. Now more and more efforts are put on the algorithms' performance. In Ref. [43], the authors first defined the concept of "asymptotic convergence factor" and "per-step convergence factor" to help measure the convergence speed. These concepts are also used in Refs. [44] and [45] to investigate the convergence speed over switching topology networks. In Ref. [46], the authors used the "decay factor" to represent the dynamics of the network topology. The largest eigenvalue of a Lyapunov-like matrix recursion is used to characterize the convergence rate of the consensus algorithm. It has been reported that the hubs are of leading role during consensus procedure. When the dominant direction is from the hub to the non-hub nodes, both the speed to reach consensus and robustness to the communication delay are greatly improved.[47] Researchers have been designing topology evolving strategies to speed up the convergence rate.[44−49] However, there still exist systems whose topology do not change very often. For instance, the power grid in the North America. The convergence speed of static topology networks still deserves a follow-up.

The purpose of the present work is to investigate the relationship between power-law distribution parameters and the consensus behaviour, mainly concerning the time cost to reach a certain level of consensus. The undirected random scale-free networks have a single connected component, without self- or multiple links. It requires to create scale-free networks which are solely determined by power-law distribution, in a stochastic way. The basic thought is simple: to create scale-free networks from power-law distributions. In the real world, different topologies may share a common degree distribution. The proposed network construction method preserves the diversity. It leads to a new path to stochastic scale-free networks of a determined degree distribution.

The following of the paper is organized as follows. In Section 2, we introduce the strategy to create networks which obey certain power-law distributions. The consensus algorithms and the measurement of consensus will be stated in Section 3. Large scale simulations and analysis are provided in Section 4. The conclusions are drawn in Section 5.

2. Scale-free network: instruction and construction

The models to create scale-free networks are widely researched. But they often bring in new parameters while offering quite limited power-law distribution parameters. The famous B-A model[1] introduces a degree parameter of the new-added vertex. Although this value has no impact on the final power-law exponent $\gamma = 3$ as time (or node number) tends to infinity, it strongly impacts the distribution if the time is limited. The dynamic model in Ref. [10] brings in a strength parameter, and the power-law distribution has $\gamma \in (2, 3]$. Although it can provide a larger interval for γ by adjusting the strength parameter, it is a tedious job to assign a local strength to each edge. It is an excellent algorithm to scale-free networks, but not to certain power-law distributions. A rewiring model which does not change the total size of the graph is introduced in Ref. [11]. It can only provide $\gamma = 2$ approximately. Under certain conditions the degree distribution would be far from a power law. The copy model[12] does not guarantee the value of power-law parameters. An evolving scale-free model[13] may create power-law degree distribution with the exponent $\gamma \in (3, \infty)$, with a new introduced parameter "attractiveness". Within the present work, an algorithm to create scale-free networks is raised. The basic idea is from a reversed thinking: creating scale-free networks from power-law distributions, rather than rediscovering power laws in ready-made networks.

Starting with the symbols and some graph theory preliminaries, the method to construct networks with given power-law distribution is introduced in this section. Two networks are given as samples for the proposed method, comparing with the classic B-A model.

2.1. Preliminaries

For a multi-agent system with N agents, the network topology can be denoted by a graph G whose adjacency matrix is A. The element of the i-th row and the j-th column in matrix A indicates the connection state between agents i and j. Assume $A = [a_{ij}]_{N\times N}$, $a_{ij} \neq 0$ if nodes i and j are connected, and j is called a neighbour of node i. All the neighbours of agent i form the set N_i. For unweighed graph, $a_{ij} = 1$ when i and j are connected. If the nodes i and j are disconnected, $a_{ij} = 0$. If the graph is undirected, $a_{ij} = a_{ji}$.

Let B be the $N \times N$ diagonal matrix, where the diagonal elements b_j is the number of neighbours the j-th agent has. Then the graph Laplacian can be defined as

$$L(G) = B - A.$$

The second smallest eigenvalue λ_2 of $L(G)$ is called the algebraic connectivity,[50] which is indicated by $\alpha(G)$. By this definition, the analysis of the consensus speed in networks can be reduced to the spectral analysis of the graph Laplacian. Let $\nu(G)$ and $\eta(G)$ denote the node-connectivity and the edge-connectivity of a graph G, we have:[46]

$$\lambda_2(L(G)) = \alpha(G) \leq \nu(G) \leq \eta(G).$$

According to this inequality, a network with a larger algebraic connectivity is more robust to both node-failures and edge-failures. Researchers have found that the algebraic connectivity could imply the convergence speed of a consensus problem.[49,51,52] Intuitively, the connection situation should have a strong relationship to the network consensus behaviour.

A necessary condition for a multi-agent system to reach consensus is that all the agents are connected.[32] A necessary and sufficient condition for the stability of the network while reaching consensus is given as[52]

$$\tau \leq \tau_{\max} = \frac{\pi}{2\lambda_N},$$

where τ is the time delay in the network and λ_N is the largest eigenvalue of the graph Laplacian. This means that λ_N can be a measurement of network robustness over delay.

2.2. Degree assignment

In general, the power-law distribution can be described by the probability density function:

$$p(x)\mathrm{d}x = \boldsymbol{Pr}(x \leq X < x + \mathrm{d}x) = Cd^{-\gamma}\mathrm{d}x.$$

This distribution diverges at $x = 0$. To avoid this, there must be a lower bound $x_{\min} > 0$ and then the density function is:

$$p(x) = \frac{\gamma - 1}{x_{\min}}\left(\frac{x}{x_{\min}}\right)^{-\gamma}. \tag{1}$$

As the edge number of a vertex is always an integer, the discrete form of probability density function is of the form

$$p(d) = \boldsymbol{Pr}(\text{degree} = d) = Cd^{-\gamma}.$$

In a connected graph, no vertex is with edge degree 0. With the lower bound $d_{\min} > 0$ on the power-law behaviour,

$$p(d) = \frac{d^{-\gamma}}{\zeta(\gamma, d_{\min})},$$

where

$$\zeta(\gamma, d_{\min}) = \sum_{n=0}^{\infty}(n + d_{\min})^{-\gamma}$$

is the generalized Hurwitz zeta function.

Considering that the formulas for continuous power-law distributions are much simpler than those for discrete distributions, it is common to approximate a discrete power-law distribution with its continuous counterpart. There are several methods to achieve this goal, one of which is to round the samples generated from continuous power law to the nearest integer. This approach could provide quite accurate results. We use this method to generate the degree values for each vertex. According to the probability density function (1), we can obtain the cumulative distribution function (CDF) for edge degree as

$$\boldsymbol{Pr}(\text{degree} \leq d) = 1 - \left(\frac{d}{d_{\min}}\right)^{1-\gamma}.$$

Let the graph order (node number) be N, build a random vector b that $b_i \in (0,1), i = 1, 2, \ldots, N$. If its elements obey the continuous uniform distribution $b_i \sim U[0,1]$, one can obtain the degree of each node by solving the above equation

$$d = \mathrm{ROUND}(d_{\min} \cdot (1 - b)^{1/(1-\gamma)}). \tag{2}$$

Here the degrees are all set to be integer in case of unweighed graph.

2.3. Building the scale-free networks

A possible solution to build a scale-free network is to make use of the degree values generated in Eq. (2). However, this method is not good because deadlocks might happen. There might be no suitable node pairs to insert a new edge. In order to build a network with given parameters, we raise the solution that can crack into the deadlocked graph and put new edges into it.

Let A represent the adjacency matrix of the network. Assume that the node number N and power-law distribution parameters $\{\gamma, d_{\min}\}$ are given, we suggest the following procedures to generate a scale-free network:

(i) Assign the degree value d to each node according to Eq. (2). Arrange d in decreasing order. If $\sum d$ is odd, $d_1 = d_1 + 1$. Let $D = d$.

(ii) Insure the spanning tree. For each nodes i except the first one, pick a node r in the set $\{1, 2, \ldots, i-1\}$, which has $D_r > 0$. Connect node i and r, $D_i = D_i - 1$, $D_r = D_r - 1$. Take step (v) to solve the deadlock if a node cannot be inserted into the present subgraph, till a connected component with N nodes is completed.

(iii) Insert edges into the network according to the degree values D. For each nodes i, while $D_i > 0$, random select a node $r \in V, r \neq i, D_r > 0$. Connect node i and r, $D_i = D_i - 1$, $D_r = D_r - 1$. In fact, the deadlocks might happen here. Leave the nodes which cannot be paired in the graph. In case the total edge number in an undirected graph is even, the sum of leftover degrees must be a positive even integer.

(iv) For each node i with extra degree larger than 2, randomly select a pair of nodes $\{p, q\}$ which are not connected to it. Break the edge pq and insert the node i, to form edges ip and iq. $D_i = D_i - 2$. See Fig. 1 for a visualized explanation.

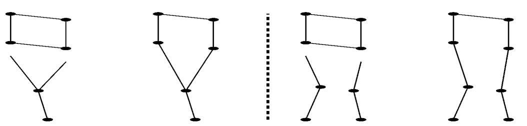

Fig. 1. To solve the deadlocks while building the graph. Left: step 4. Right: step 5.

(v) After the above step, there are some nodes with extra degree equals 1. For 2 nodes $\{i, j\}$ with extra degree, randomly select a pair of nodes $\{p, q\}$ which are not connected to either of $\{i, j\}$. The present subgraph should keep connected if edge pq was eliminated. Break edge pq, add 2 edges ip and jq. $D_i = D_i - 1$,$D_j = D_j - 1$.

The provided method ensures the diversity of the generated graphs since the nodes and edges are chosen stochastically. Like the other algorithms, there are limitations to the proposed method. Since the degree value of a node in unweighed graphs must be an integer, the exponent cannot be continuous. But the proposed method can provide close-enough samples for a power-law distribution. Since the edge number of a graph will of course be on $[n-1, n(n-1)/2]$, the graph size will determine the limitation of power law parameters. The algorithm is demonstrated to be capable of solving most of the deadlocks.

The parameters are estimated with maximum likelihood estimation (MLE), and then tested by Kolmogorov–Smirnov test.[53] The goodness-of-fit is represented by the maximum distance between sample data and the fitted function. It is referred to as "accuracy of fitness" in the present work. Its definition is:

$$acc = \max_{d \geq d_{\min}} |F(d) - P(d)|,$$

where $F(d)$ is the CDF of the sample data and $P(d)$ is the CDF of the estimated power-law distribution.

The first 2 samples in Fig. 2 are networks generated by the above method. The desired distributions are $p(d) = d^{-3}/\zeta(3, 7)$ and $p(d) = d^{-4}/\zeta(4, 10)$. The estimations are quite close to the desired values. Sample 3 is a 5000-node network from the B-A model, with a "seed" of 15-node full graph, and degree 7 for each new attached node. But the estimated parameters are $\hat{\gamma} = 2.8510, \hat{d}_{\min} = 8$, which are far from the claimed $\gamma = 3$ and desired $d_{\min} = 7$.

Another issue about the B-A model is the average clustering coefficient. The networks from the B-A model always have average clustering coefficient near 0. But this may not be true for all the scale-free networks. The proposed algorithm has the average clustering coefficient varying in a wider area (Fig. 3). The clustering coefficient tends to 0 as γ grows, for the rapid decreasing power law indicates star-like networks. The average clustering coefficients are between 0.4127 and 0.8529 when $\gamma = 2$ and $d_{\min} = 5$, due to the simulation data.

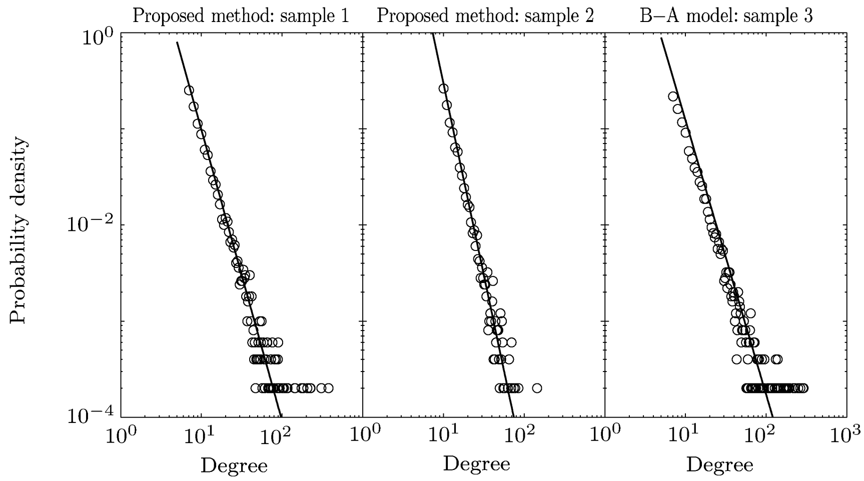

Fig. 2. Simulations on proposed algorithm of 5000 nodes. Circles denote the samples. Lines represent the probability density functions obtained via maximum likelihood estimation (MLE). Sample 1: constructed by the proposed method. The desired power-law distribution is $p(d) = d^{-3}/\zeta(3,7)$. The data fitting results are $\hat{\gamma} = 3.0230, \hat{d}_{\min} = 7, acc = 0.0071$. Sample 2: constructed by the proposed method. The desired power-law distribution is $p(d) = d^{-4}/\zeta(4,10)$. The data fitting results are $\hat{\gamma} = 3.9770, \hat{d}_{\min} = 10, acc = 0.0075$. Sample 3: constructed by B-A model. The desired power-law distribution is $p(d) = d^{-3}/\zeta(3,7)$. The data fitting results are $\hat{\gamma} = 2.8510, \hat{d}_{\min} = 8, acc = 0.0086$.

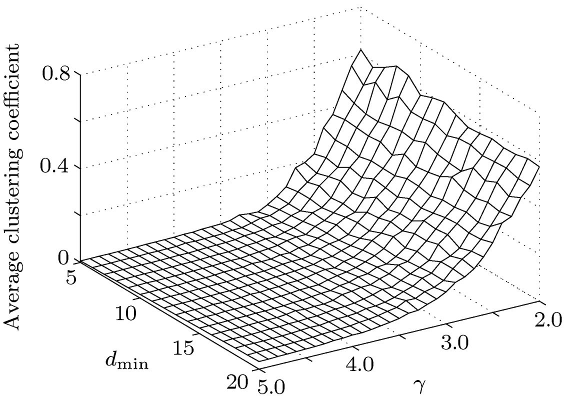

Fig. 3. The average clustering coefficient of scale-free networks constructed by the proposed method.

3. Consensus protocols

Two consensus protocols are taken in the paradigms, to investigate the convergence speed of networks. One of them is continuous, and the other is discrete. It is necessary to point out that the discrete one is not the counterpart of the continuous algorithm. Let $h_i(t)$ indicate the state of the i-th agent at time t, thus $h(t) = \{h_1, h_2, \ldots, h_N\}$ is the state of the whole network. Assume N_i denotes the set of neighbours of node i. The continuous consensus algorithm[29] could be presented as:

$$\dot{h}_i(t) = -\sum_{j\in N_i} L_{ij} h_j(t). \tag{3}$$

Assume d_i indicates the degree of the node n_i. If $h_i(k)$ is the state of node n_i at the step k, the discrete consensus algorithm[31] could be represented as:

$$h_i(k+1) = \frac{1}{d_i + \sum_{j\in N_i} d_j} \times \left(h_i(k) d_i + \sum_{j\in N_i} h_j(k) d_j \right). \tag{4}$$

A measurement of the "disagreement" among the agents could determine whether the consensus has been reached.[31] It is defined as:

$$S(h(t)) = \sum_i S_i(h(t)) = \sum_i \sum_{j\in N_i} \|h_i(t) - h_j(t)\|. \tag{5}$$

For an arbitrary positive value ϵ, if there exists a t_c so that when $t > t_c$, $S(t) < \epsilon$, the system is said to reach ϵ-consensus at t_c. The symbol t_c is used as the consensus time within this paper for convenience. It is proved that the control laws presented in Eqs. (3) and (4) are both convergent. Therefor $S(h(t))$ converges to zero as $t \to \infty$. The two protocols can both reach an ϵ-consensus state with arbitrary small ϵ.

4. Convergence speed of scale-free networks

The node number N of the networks is set to 1000. In fact, it is just a limit of the generated graph order.

It could be any positive integer, which does not determine the power-law parameters. Thirty-two samples are generated for each set of parameters $\{\gamma, d_{\min}\}$. γ varies between 2.1 to 5 with the interval of 0.1, and the integer $d_{\min}$ varies between 5 to 20. The averaged results of 32 samples are stated in this section. The initial state of nodes in each run is set to be the same as $h_i(0) = \mathrm{rand}\,(0,1)$.

The power-law parameters γ and $d_{\min}$ describe scale-free behaviour. The major purpose is to find out if these parameters influence time cost to reach ϵ-consensus. The investigation on algebraic connectivity is conducted since it can be seen as a measurement of convergence speed. The robustness over time delay is discussed by the study of λ_N. The observations on graph size is also included. Two consensus protocols are the continuous algorithm (3) and the discrete one (4).

4.1. Time expense and the power-law distribution

The first investigation is about ϵ-consensus time t_{c}. See Fig. 4. The consensus time t_{c} ($\epsilon = 10^{-3}$) grows dramatically as the γ increases. Meanwhile, the network reaches consensus more quickly when the minimum degree $d_{\min}$ is larger. This is because the larger $d_{\min}$ results in larger graph size, which could lead to better connectivity of the graph. Although there are conclusions that too many connections in a graph may postpone the consensus, the size of a scale-free network is far from the size of the corresponding full graph. Thus the minimum degree has strong impact on the time cost. The relationship could be of the form $t_{\mathrm{c}} = {d_{\min}}^{g(\gamma)} \cdot f(\gamma)$.

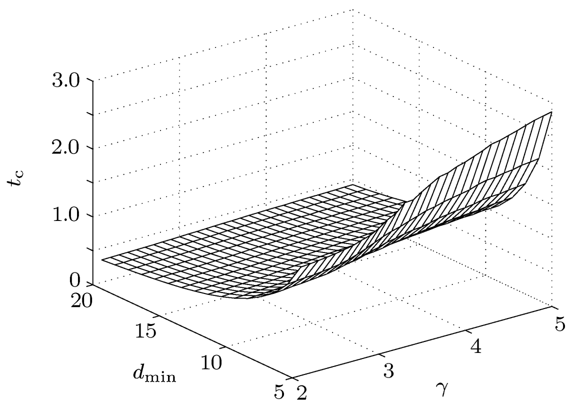

Fig. 4. Time cost t_{c} to reach ϵ-consensus. $\epsilon = 0.001$.

An interesting finding is that the time cost t_{c} is a reversed power-law function of the algebraic connectivity. See Fig. 5. The larger the λ_2 is, the sooner the consensus could be achieved. The $t_{\mathrm{c}} = C \cdot \lambda_2^{\beta}$ relationship is very similar with each other in spite of different values of γ. That is to say, the consensus behaviour of scale-free networks could be characterized with the algebraic connectivity, besides the consensus protocol. As the consensus algorithm is the external cause, the algebraic connectivity could be the only character of a network when discussing the consensus problems. The estimated function for the data in Fig. 5 is $t_{\mathrm{c}} = 3.7493\lambda_2^{-0.8326}$, with the sum of squares due to error equal to 0.3812.

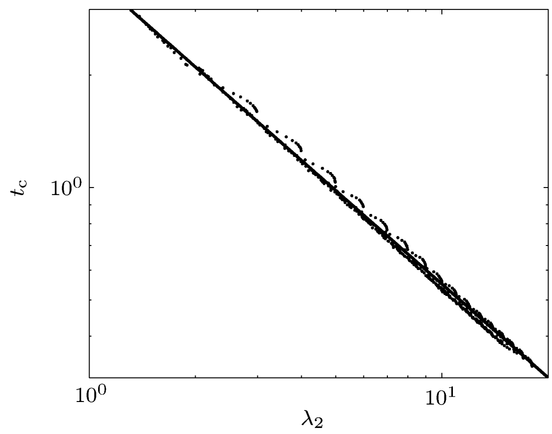

Fig. 5. Time cost versus algebraic connectivity, with different values of γ. The line indicates the estimation with parameters $\hat{C} = 3.7493, \hat{\beta} = -0.8326$.

4.2. Algebraic connectivity and the power-law distribution

See Fig. 6 for the relationship among λ_2, $d_{\min}$ and γ. The λ_2 decreases as γ increases. The relationship between λ_2 and $d_{\min}$ is perfectly monotonically increasing. Actually, they fit very well to a linear relationship. An approximated value for λ_2 is $d_{\min}$,[26] and there is a linear relationship between them. In fact, the algebraic connectivity cannot be solely determined by the degree distribution.[27] It is related to the network topology. As γ decreases, there is a saturation-like phenomenon that limits the increase of λ_2. Obviously, γ has much smaller impact on λ_2 than $d_{\min}$ does. The relationship among these variables could be of the form $\lambda_2 = d_{\min} \cdot f(\gamma)$.

Figure 7 presents the variance of the λ_2 at each sampling point in Fig. 6. The reading of Fig. 7 is the diversity of the network topology. When γ is near 5, the graph size is relatively small (see Subsection 4.3), so that the networks are star-like and similar to each other. While near $\gamma = 2.1$, the graph size is too large to offer the diversity. The variance is larger in the middle part of the figure, which means the algebraic varies more often in this region. The interpretation is that the networks of the same power-law distribution may have quite different topologies.

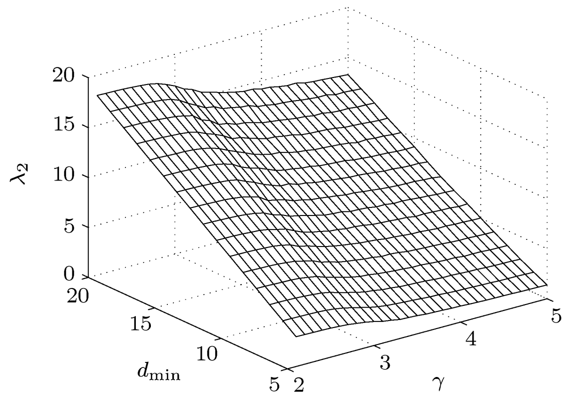

Fig. 6. Algebraic connectivity versus power-law distribution parameters.

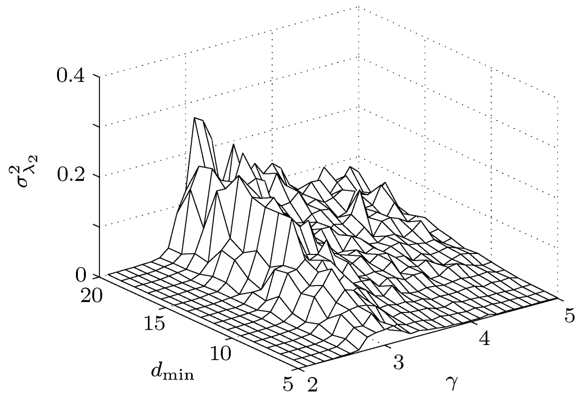

Fig. 7. The variance of algebraic connectivity.

4.3. Graph size $|E(G)|$

The convergence speed is related to both the graph size and the graph topology. Figure 8 shows how γ and $d_{\min}$ influence the graph size $|E(G)|$. $|E(G)|$ is monotonically increasing with γ, as well as with $d_{\min}$. The relationship between $|E(G)|$ and $d_{\min}$ fits very well with straight lines. According to the degree distribution, $b \sim U[0,1]$. All the elements in vector b have the expected value $E(b_i) = 0.5$. As a result, $E(d_i) = d_{\min} \cdot 0.5^{1/(1-\gamma)}$. Since they are independent of each other, the expected value of the total edge number of the network is

$$E(|E(G)|) = E(\mathrm{tr}(L)) = E\left(\sum d_i\right) = \sum E(d_i) = \sum 0.5^{1/(1-\gamma)} \cdot d_{\min} = 0.5^{1/(1-\gamma)} \cdot N \cdot d_{\min}. \quad (6)$$

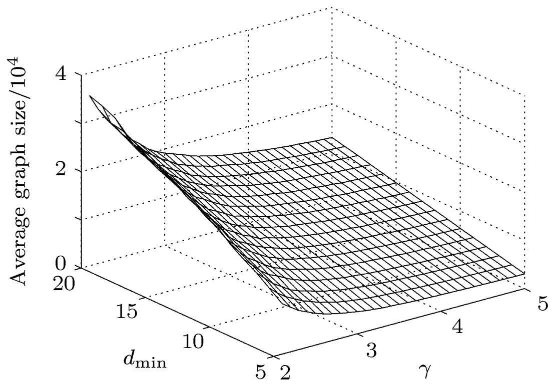

Fig. 8. Graph size versus power-law distribution parameters.

4.4. The behaviour of λ_N and eigenratio

The λ_N measures the robustness of a network with respect to delays. The main question is that whether the attempts trying to increase convergence speed lead to a considerable decrease in robustness over time delay. In Fig. 9, although the data are quite noisy, one can still tell that the increase in γ leads to dramatic decrease in λ_N. The change of λ_N is similar to that of λ_2. It means that the robustness over time delay declines while the robustness over node or edge failures increases.

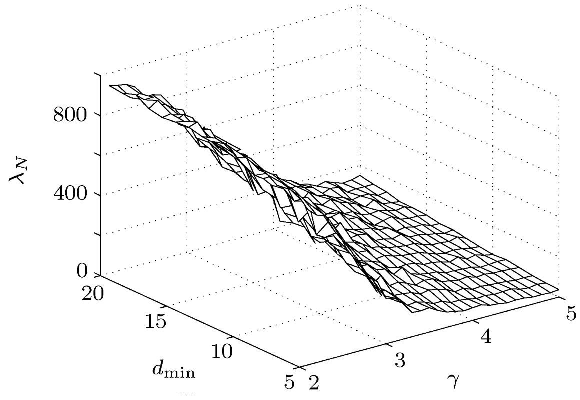

Fig. 9. The λ_N versus power-law distribution parameters.

As stated in literatures, the largest eigenvalue of the graph Laplacian can be approximated by $d_{\max} + 1$.[10] The symbol $d_{\max}$ denotes the maximum degree value of a node. Due to the power-law distribution, the average of $d_{\max}$ could be obtained as

$$\langle d_{\max}\rangle \simeq d_{\min} N^{1/(\gamma-1)} \exp\left[\frac{d_{\min}^{\gamma-1}}{N^{\gamma-2}}\right] \Gamma\left[\frac{\gamma-2}{\gamma-2}, \frac{d_{\min}^{\gamma-1}}{N^{\gamma-2}}\right],$$

where Γ is the incomplete Gamma function. It follows the approximation and the simulation that γ has much stronger influence on λ_N than $d_{\min}$ does.

The eigenratio is defined as λ_N/λ_2. Since the second smallest and the largest eigenvalues have similar performance, it is necessary to compare them in a reasonable way. The eigenratio is a measurement of synchroniability of a network. In Fig. 10 it is observed that large eigenratio shows up with small γ and $d_{\min}$.

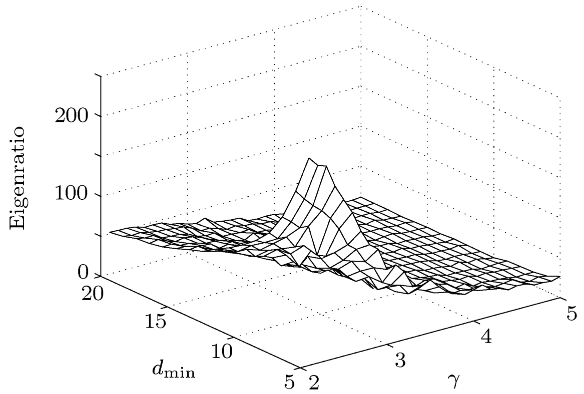

Fig. 10. The eigenratio of scale-free networks.

4.5. The performance of the discrete consensus algorithm

The investigation on discrete control law (4) has led to very similar results. The initial states are set randomly between 0 and 100. The iterations k_c taken to reach the ϵ-consensus holds roughly the same shape as t_c does (Fig. 11). See Fig. 12 for the relationship between iterations and algebraic connectivity. The estimated parameters of the reversed power-law function are $k_c = 32.5586\lambda_2^{-0.3958}$. Other relations studied for continuous algorithm (3) hold similar conclusions with their counterparts for the discrete algorithm (4).

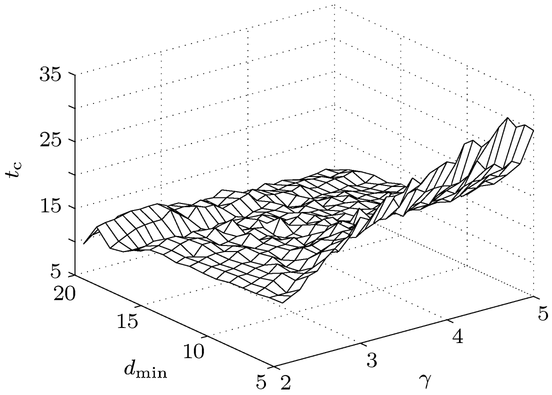

Fig. 11. Iterations k_c taken to reach ϵ-consensus while using the discrete consensus algorithm (4). $\epsilon = 0.1$.

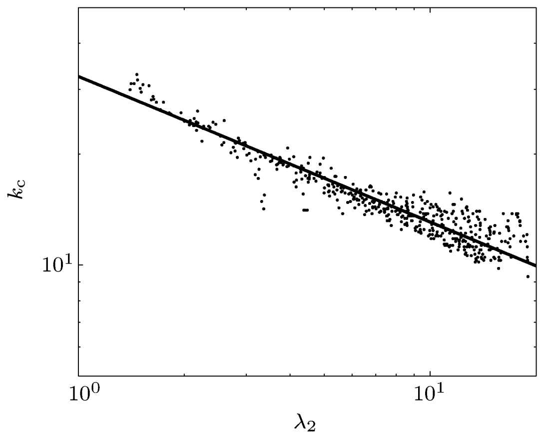

Fig. 12. Iterations versus algebraic connectivity while using the discrete consensus algorithm (4). The estimated power-law parameters are $\hat{C} = 32.5586$, $\hat{\beta} = -0.3958$.

5. Conclusion

The relationship between convergence speed of consensus behaviour and scale-free network parameters is studied in this paper. Both continuous and discrete linear consensus protocols are discussed, which lead to similar conclusions. The time expense t_c to reach ϵ-consensus is high when the power-law distribution parameter γ is large, or the minimum degree of each node $d_{\min}$ is small. The algebraic connectivity decreases while γ grows, and is approximated by $d_{\min}$. Time cost t_c is an reversed power-law function of λ_2, as well as the iterations k_c to reach ϵ-consensus. The algebraic connectivity could be viewed as the internal character of networks on consensus behaviour. It is demonstrated that the robustness over time delays declines while the robustness over node and edge failures increases.

Besides the investigation on consensus behaviour, a construction scheme for scale-free networks due to given power-law distribution is introduced. It follows a reversed thinking: creating the networks from the power laws, rather than fitting the networks into power laws. The proposed method can provide close-enough samples with desired power-law distributions.

The future work includes the behaviour of nonlinear consensus protocols over dynamic networks. The consensus on graphs with positive and negative connection strength will be interesting since such cases are the real models of the nature and human society.

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