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Adaptive Dynamic Surface Based Nonsingular Fast Terminal Sliding Mode Control for Semistrict Feedback System

This paper focuses on an adaptive dynamic surface based nonsingular fast terminal sliding mode control (ADS-NFTSMC) for a class of nth-order uncertain nonlinear systems in semistrict feedback form. A simple and effective controller has been obtained by introducing dynamic surface control (DSC) technique on the basis of second-order filters that the "explosion of terms" problem caused by backstepping method can be avoided. The nonsingular fast terminal sliding mode control is adopted in the last step of the controller design, and the error convergence rate is improved. An composite adaptive law is used to gain fast and accurate parameter estimation. Finally, simulation results are presented to illustrate the effectiveness of the proposed method. [DOI: 10.1115/1.4005373]

1 Introduction

Control of uncertain systems has been the focus of the control realm. Robust control and adaptive control are two commonly used methods, which can solve uncertain issues in control systems. Sliding mode control (SMC) which provides invariance to matched uncertainties once the system dynamics are controlled on the sliding mode is an efficient and effective robust approach to manipulate control problems of uncertain systems. It has been widely used in practical systems for its simplicity in controller design [1]. However, the invariance of SMC is lost in systems with mismatched uncertainties. Adaptive backstepping control is a systematic design technique that can tackle mismatched uncertainties in systems, which can be transformed into parameter strict feedback form or semistrict feedback form [2]. The combination of these two methods may help to overcome some problems of either controller and has been widely studied in the last few years. An adaptive backstepping sliding mode control (ABSMC) approach is proposed in Ref. [3] for systems with relative degree one, and then the method is extended to parameter pure feedback system [4] and parameter strict feedback system [5]. In Ref. [6], a systematic design method of ABSMC for semistrict feedback system is discussed, and the system states can be steered onto the sliding mode. A symbolic algebra toolbox allows straight forward design is presented by Zinober and Rios-Bolivar [7]. Some applications to linear induction motor is given by Lin et al. [8,9], where the bounds of the lumped uncertainty is estimated by adaptive laws and good transient response is ensured by SMC.

Although the ABSMC have many merits, but a drawback of the design procedure is the "explosion of terms" caused by backstepping approach, and the dynamic surface control (DSC) technique has been proposed to avoid this problem. A first order low-pass filter is used to acquire the derivative information of the virtual control at each step of the conventional backstepping design procedure in Refs. [10]–[16]. Yip and Hedrick proved the semiglobal stability of the combined adaptive backstepping-first order filter system by singular perturbation approach [10] and applications on automobile control were given in Ref. [11]. The robust control for non-Lipschitz system is given by Swaroop et al. [12] where arbitrary small tracking error can be achieved. Song et al. [13] proposed a systematic approach to design and to analyze the controller gains and filter time constants for DSC via convex optimization. The DSC method is then extended to radial basis function (RBF) neural network based adaptive control system [14,15] where uncertain nonlinearities are approximated by RBF networks. In Ref. [16], the composite adaptive law is introduced to the DSC method to enhance the parameter estimation. However, the measurement noise can directly propagate to the derivative information of the virtual control effort with the first order filter [17]. In Ref. [17], Lu et al. indicate that second-order filters have some advantages over the first order filter in DSC, and researchers introduce the second-order filter based DSC to several applications, such as fin stabilizer [18], mobile manipulators [19], and flight controller [20].

One drawback of the aforementioned control methods is only the linear sliding surface is used. To improve the error convergence rate, nonsingular fast terminal sliding mode control (NFTSMC) [21], which can achieve finite-time stability is adopted in the last step of ABSMC. In Ref. [22], an adaptive backstepping based terminal sliding mode controller for parameter strict feedback system is proposed, where the finite-time convergence of the error in the last step is achieved. In Ref. [23], a robust and adaptive terminal sliding mode control based on backstepping is presented for strict-block-feedback systems with uncertainties. The introduction of NFTSMC can enhance the performance of the backstepping based control system in comparison with ABSMC. But NFTSMC cannot deliver the same convergence performance, while the system states are far away from the equilibrium point.

In this paper, motivated by the pioneering works reported in the literature, the above mentioned problems are addressed. An adaptive dynamic surface based nonsingular fast terminal sliding mode control (ADS-NFTSMC) method is discussed for a class of nth-order semistrict feedback systems. The design is performed in a step by step manner. For each step, a second-order filter is adopted to gain the information of derivative of the virtual control effort. An adaptive robust virtual control law is designed to stabilize each subsystem in the former $n-1$ steps. While an adaptive NFTSMC is employed in the last step to stabilize the whole system. NFTSMC provides fast finite-time convergence of errors either far away from or near by the equilibrium point, and the performance of the

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close-loop system can be improved. The uncertain parameters in the system are updated by composite adaptive law, which can achieve fast and accurate parameter estimate.

This paper is organized as follows. In Sec. 2, the problem and some preliminaries are presented. Design procedures of the controller and parameter adaptive law are given in Sec. 3. In Sec 4, stability analysis of the system is declared. In Sec. 5, the advantages of the proposed ADS-NFTSMC are illustrated by some comparative simulations, and some conclusions are drawn in Sec. 6.

2 Problem Formulation and Preliminaries

In this section, the control problem is formulated. Consider the following semistrict feedback uncertain system [16]

$$\begin{cases} \dot{x}_i = -\varphi_i^{\mathrm{T}}(\varpi_i)\theta_i^0 + b_i x_{i+1} + \Delta_i(x,t) \\ \qquad i = 1,2,\ldots,n-1 \\ \dot{x}_n = -\varphi_n^{\mathrm{T}}(\varpi_n)\theta_n^0 + b_n u + \Delta_n(x,t) \\ y = x_1 \end{cases} \tag{1}$$

where $\varpi_i = [x_1,\ldots,x_i]^T \in R^i$, $\varpi_n = x = [x_1,\ldots,x_n]^T \in R^n$ is the system state vector, $y \in R$ is the system output, and $u \in R$ is the system input. In the ith $(i = 1,2,\ldots,n)$ channel, b_i is the unknown input gain, $\theta_i^0 \in R^{m_i}-1$ is the uncertain parameter vector, $\varphi_i(\bar{\omega}_i) \in R^{m_i-1}$ is smooth functions, which is known and $\Delta_i(x,t)$ is unmodeled dynamics.

As shown in system (1), the parameters of the model cannot be accurately determined, but we assume that the uncertain parameters lie in some previously known intervals, as shown in assumptions 1 and 2. In addition, assumption 3 is for the desired trajectories.

Assumption 1. *The uncertain parameters θ_i^0 and b_i are bounded and the bounds are known in advance, i.e.*

$$\begin{aligned} &\theta_i^0 \in \Omega_{\theta_i^0} := \{\theta_i^0 : \theta_i^{0,\min} \le \theta_i \le \theta_i^{0,\max}\} \\ &b_i \in \Omega_{b_i} := \{b_i : 0 < b_i^{\min} \le b_i \le b_i^{\max}\} \end{aligned} \tag{2}$$

where $\theta_i^{0,\min}$, $\theta_i^{0,\max}$, $b_i^{\min}$, and $b_i^{\max}$ are known. Furthermore, $b_i^{\min} > 0$, which conforms to the physical point of view.

Assumption 2. *The unmodeled dynamics is bounded, i.e., $\Delta_i \in \Omega_{d_i} := \left\{\Delta_i : |\Delta_i| \le \varsigma_{d_i}\right\}$, where $\varsigma_{d_i} > 0$ is known.*

Assumption 3. *The desired trajectory x_d is continuous, and its first-order derivative $\dot{x}_d$ and second-order derivative $\ddot{x}_d$ are bounded and available.*

In order to design the controller conveniently, we transform the model in Eq. (1) into the following form:

$$\begin{cases} \theta_{i,m_i}\dot{x}_i = -\varphi_i^{\mathrm{T}}(\varpi_i)\bar{\theta}_i + x_{i+1} + \Delta_i'(x,t) \\ \qquad i = 1,2,\ldots,n-1 \\ \theta_{i,m_n}\dot{x}_n = -\varphi_n^{\mathrm{T}}(\varpi_n)\bar{\theta}_n + u + \Delta_n'(x,t) \\ y = x_1 \end{cases} \tag{3}$$

where $\theta_{i,m_i} = \frac{1}{b_i}$, $\bar{\theta}_i = \frac{\theta_i^0}{b_i}$, $\theta_i \in R^{m_i-1}$, $\Delta_i'(x,t) = \frac{\Delta_i(x,t)}{b_i}$, $i = 1,\ldots,n$.

According to assumption 1, there exists $\bar{\theta}_i^{\min}$, $\bar{\theta}_i^{\max}$, $\theta_{i,m_i}^{\min}$, and $\theta_{i,m_i}^{\max}$ satisfying $\bar{\theta}_i \in [\bar{\theta}_i^{\min}, \bar{\theta}_i^{\max}]$ and $\theta_{i,m_i} \in \left[\theta_{i,m_i}^{\min}, \theta_{i,m_i}^{\min}\right]$, respectively, where $i = 1,\ldots,n$. Moreover, $\bar{\theta}_i^{\min}$, $\bar{\theta}_i^{\max}$, $\theta_{i,m_i}^{\min}$, and $\theta_{i,m_i}^{\max}$ are functions of $\theta_i^{0,\min}$, $\theta_i^{0,\max}$, $b_i^{\min}$, and $b_i^{\max}$ in respect. From assumption 2, it can be seen that $\Delta_i'(x,t)$ is bounded. So the model in Eq. (3) has the following properties:

Property 1

$$\begin{aligned} &\bar{\theta}_i \in \Omega_{\theta_i} := \{\bar{\theta}_i : \bar{\theta}_i^{\min} \le \bar{\theta}_i \le \bar{\theta}_i^{\max}\} \\ &\theta_{i,m_i} \in \Omega_{\theta_{i,m_i}} := \{\theta_{i,m_i} : 0 < \theta_{i,m_i}^{\min} \le \theta_{i,m_i} \le \theta_{i,m_i}^{\max}\} \\ &\Delta_i' \in \Omega_{\Delta_i'} := \{\Delta_i' : |\Delta_i'| \le \varsigma_{d_i}'\} \end{aligned} \tag{4}$$

where $\bar{\theta}_i^{\min}$, $\bar{\theta}_i^{\max}$, $\theta_{i,m_i}^{\min}$, $\theta_{i,m_i}^{\max}$, and ς_{d_i}' are known.

Consider model (3), which has unknown parameters, disturbance and unmodeled dynamics, the control problem is formulated as follows. Given the desired trajectory x_d, the object is to design a bounded control law for the input u such that the system is stable and the output y tracks the desired output trajectory x_d as closely as possible in spite of the aforementioned uncertainties.

Some notions that will be used in the following sections are introduced. For simplicity, the following notations will be used: $\bullet_i$ for the ith component of the vector $\bullet$, $\hat{\bullet}$ for the estimate of $\bullet$, $\bullet_{\min}$ for the minimum value of $\bullet$ and $\bullet_{\max}$ for the maximum value of $\bullet$. $\|\bullet\|$ is the Euclidean norm of $\bullet$. $\lambda_{\min}(*)$ and $\lambda_{\max}(*)$ are the minimum eigenvalue and maximum eigenvalue of the matrix $*$, respectively. The operation $\ge$ (or $\le$) for two vectors is performed in terms of the corresponding elements.

3 Controller Design

In this section, the DSC technique based on second-order filters is used to design an ASMC instead of the integral backstepping method, and in sequence the explosion of terms problem is avoided. Then, a composite adaptive law is introduced to estimate the uncertain parameters.

3.1 Control Law Design. A dynamic surface control approach on the basis of second-order filters is adopted in this paper. The design is based on the following error variables:

$$\begin{cases} e_1 = x_1 - x_d \\ e_2 = x_2 - z_{11} \\ \quad \vdots \\ e_n = x_n - z_{(n-1)1} \end{cases} \tag{5}$$

where x_d is the desired trajectory of the system and the first step, while $z_{(i-1)1}$ is the desired trajectory for the ith step, $z_{(i-1)1}$ is the output of the filter with input α_{i-1}, and α_{i-1} is the virtual control in the $i-1$ step.

The design process is as follows:

Step 1. The error variables related to the design in step 1 are $e_1 = x_1 - y_d$ and $e_2 = x_2 - z_{11}$. The derivative of e_1 is $\dot{e}_1 = \dot{x}_1 - \dot{y}_d$. Then, from Eq. (3), we can obtain the following dynamics:

$$\begin{aligned} \theta_{1,m_1}\dot{e}_1 &= \theta_{1,m_1}\dot{x}_1 - \theta_{1,m_1}\dot{y}_d \\ &= -\varphi_1^{\mathrm{T}}(\varpi_1)\bar{\theta}_1 + x_2 - \Delta_1'(x,t) - \theta_{1,m_1}\dot{y}_d \\ &= x_2 - \Psi_1^{\mathrm{T}}(\varpi_1)\Theta_1 - \Delta_1'(x,t) \end{aligned} \tag{6}$$

where $\Psi_1(\varpi_1) = [\varphi_1^{\mathrm{T}}(\varpi_1), \dot{y}_d]^{\mathrm{T}}$, $\Theta_1 = [\bar{\theta}_1^{\mathrm{T}}, \theta_{1,m_1}]^{\mathrm{T}}$, and $\Theta_1 \in R^{m_1}$.

If x_2 is the control input in the first channel of model (3), then from Eq. (6) one could synthesize a virtual control law α_1 for it. A similar controller as that in Ref. [24] for Eq. (6) is as follows:

$$\alpha_1 = \alpha_{1a} + \alpha_{1s} \tag{7}$$

where α_{1a} is the adaptive control term and α_{1s} is the robust control term. Let

$$\alpha_{1a} = \Psi_1^{\mathrm{T}}(\varpi_1)\hat{\Theta}_1 \tag{8}$$

where $\hat{\Theta}_1 = [\hat{\bar{\theta}}_1^{\mathrm{T}}, \hat{\theta}_{1,m_1}]^{\mathrm{T}}$.

The robust control term α_{1s} contains two parts are

$$\begin{aligned} &\alpha_{1s} = \alpha_{1s,1} + \alpha_{1s,2} \\ &\alpha_{1s,1} = -k_{11}e_1 \\ &\alpha_{1s,2} = -k_{12}\tanh(\hbar_1 k_{12} e_1) \end{aligned} \tag{9}$$

where $k_{11} > 0$, $k_{12} > \varsigma_{d_1}'$, $\hbar_1 > 0$, and "tanh" is the hyperbolic tangent function.

In traditional backstepping based sliding mode control, the derivative of α_1 is needed in next step; however, it is difficult to find the derivatives. In this paper, filters are used to eliminate the analytic computation, the derivative of the virtual control effort. In step 1, the proposed filter which is similar to that in Refs. [19] and [20] is as follows:

$$\dot{z}_{11} = -\frac{1}{\tau_{11}}(z_{11} - \alpha_1) - g_{11}\tanh(\lambda_{11}g_{11}(z_{11} - \alpha_1)) - e_1 \quad (10)$$

where $\tau_{11} > 0$, $g_{11} > 0$, $\lambda_{11} > 0$. τ_{11} is the time constant of the filter, g_{11} and λ_{11} are designed constants.

The filter dynamics should be considered in stability analysis as the filter is used, and the Lyapunov candidate is defined as

$$V_1 = V_{e_1} + V_{\tilde{\Theta}_1} + V_{h_{11}} \quad (11)$$

where $V_{e_1} = \frac{1}{2}\theta_{1,m_1}e_1^2$, $V_{\tilde{\Theta}_1} = \frac{1}{2}\tilde{\Theta}_1^{\mathrm{T}}\Gamma_1^{-1}\tilde{\Theta}_1$, $\tilde{\Theta}_1 = \hat{\Theta}_1 - \Theta_1$, $V_{h_{11}} = \frac{1}{2}h_{11}^2$, $h_{11} = z_{11} - \alpha_1$. The derivative of V_1 is $\dot{V}_1 = \theta_{1,m_1}e_1\dot{e}_1 + \tilde{\Theta}_1^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\Theta}}_1 + h_{11}\dot{h}_{11}$.

According to Eq. (10), one can get that

$$\dot{h}_{11} = -\frac{1}{\tau_{11}}h_{11} - g_{11}\tanh(\lambda_{11}g_{11}h_{11}) - e_1 - \dot{\alpha}_1 \quad (12)$$

From Eqs. (6), (12) and noting $x_2 = \alpha_1 + e_2 + h_{11}$, it holds that

$$\begin{aligned}\dot{V}_1 &= e_1\left(\alpha_1 + e_2 + h_{11} - \Psi_1^{\mathrm{T}}(\varpi_1)\Theta_1 - \Delta_1'(x,t)\right) + \tilde{\Theta}_1^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\Theta}}_1 \\ &\quad + h_{11}\left(-\frac{1}{\tau_{11}}h_{11} - g_{11}\tanh(\lambda_{11}g_{11}h_{11}) - e_1 - \dot{\alpha}_1\right) \\ &= e_1\left(\Psi_1^{\mathrm{T}}(\varpi_1)\tilde{\Theta}_1 + \alpha_{1s} + e_2 - \Delta_1'(x,t)\right) + \tilde{\Theta}_1^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\Theta}}_1 \\ &\quad + h_{11}\left(-\frac{1}{\tau_{11}}h_{11} - g_{11}\tanh(\lambda_{11}g_{11}h_{11}) - \dot{\alpha}_1\right)\end{aligned} \quad (13)$$

If we choose the adaptive law as $\dot{\hat{\Theta}}_1 = -\Gamma_1\Psi_1(\varpi_1)e_1$ and substitute α_{1s} into Eq. (13), then

$$\begin{aligned}\dot{V}_1 &= -k_{11}e_1^2 - (k_{12}e_1\tanh(\hbar_1 k_{12}e_1) + \Delta_1'(x,t)e_1) + e_1e_2 \\ &\quad - \frac{1}{\tau_{11}}h_{11}^2 - (g_{11}h_{11}\tanh(\lambda_{11}g_{11}h_{11}) + \dot{\alpha}_1h_{11}) \\ &\le -k_{11}e_1^2 - (k_{12}e_1\tanh(\hbar_1 k_{12}e_1) - |\Delta_1'(x,t)|\cdot|e_1|) + e_1e_2 \\ &\quad - \frac{1}{\tau_{11}}h_{11}^2 - (g_{11}h_{11}\tanh(\lambda_{11}g_{11}h_{11}) - |\dot{\alpha}_1|\cdot|h_{11}|)\end{aligned} \quad (14)$$

According to Lemma A1, if $k_{12} > |\Delta_1'(x,t)|$, $g_{11} > |\dot{\alpha}_1|_{\max}$, we can get

$$\begin{aligned}&-k_{12}e_1\tanh(\hbar_1k_{12}e_1) + |\Delta_1'(x,t)|\cdot|e_1| \le \kappa/\hbar_1 \\ &-g_{11}h_{11}\tanh(\lambda_{11}g_{11}h_{11}) + |\dot{\alpha}_1|\cdot|h_{11}| \le \kappa/\lambda_{11}\end{aligned}$$

and then the derivative of V_1 satisfies

$$\dot{V}_1 \le -k_{11}e_1^2 - \frac{1}{\tau_{11}}h_{11}^2 + \varepsilon_1 + e_1e_2 \quad (15)$$

where $\varepsilon_1 = \kappa/\hbar_1 + \kappa/\lambda_{11}$. The term e_1e_2 in will be canceled in next step.

Step i. The error variables related to the design in ith step are $e_i = x_i - z_{(i-1)1}$ and $e_{i+1} = x_{i+1} - z_{i1}$. The derivative of e_i is $\dot{e}_i = \dot{x}_i - \dot{z}_{(i-1)1}$. Then, from Eq. (3), we can obtain the following dynamics:

$$\begin{aligned}\theta_{i,m_i}\dot{e}_i &= \theta_{i,m_i}\dot{x}_2 - \theta_{i,m_i}\dot{z}_{(i-1)1} \\ &= -\psi_i^{\mathrm{T}}(\varpi_i)\bar{\theta}_i + x_{i+1} - \Delta_i'(x,t) - \theta_{i,m_i}\dot{z}_{(i-1)1}\end{aligned} \quad (16)$$

Similar to step 1, if x_{i+1} is the control input of the ith channel in Eq. (3), then we can synthesize a virtual control law α_i for it. However, in Eq. (16), the derivative of $z_{(i-1)1}$ is appeared, and the following filters that are similar to those in Refs. [19] and [20] are used to approximate $\dot{z}_{(i-1)1}$.

$$\begin{cases}\dot{z}_{(i-1)2} = -\dfrac{1}{\tau_{(i-1)2}}(z_{(i-1)2} - \dot{z}_{(i-1)1}) - e_i \\ \qquad\qquad - g_{(i-1)2}\tanh\left(\lambda_{(i-1)2}g_{(i-1)2}(z_{(i-1)2} - \dot{z}_{(i-1)1})\right) \\ \dot{z}_{i1} = -\dfrac{1}{\tau_{i1}}(z_{i1} - \alpha_i) - g_{i1}\tanh(\lambda_{i1}g_{i1}(z_{i1} - \alpha_i)) - e_i\end{cases} \quad (17)$$

where $\tau_{(i-1)2}$ and τ_{i1} are the time constants of the filters, $g_{(i-1)2}$, g_{i1}, $\lambda_{(i-1)2}$, and λ_i1 are designed parameters. The first equation in Eq. (17) is used to approximate $\dot{z}_{(i-1)1}$, and the second equation in Eq. (17) is used for the design in next step.

Remark 1. In Eq. (17), the derivative of α_{i-1} is approximate by a second-order filter, which is composed of two first-order filters as that in Refs. [19] and [20]. The measurement noise can be eliminated in the virtual control effort with these filters [17].

Remark 2. A saturation compensator in Refs. [19] and [20] is used to achieve fast transient response of each filter. In Eq. (17), the hyperbolic tangent function, which is superior to the saturation compensator is used, and the performances of the filters can be improved.

Let $h_{(i-1)2} = z_{(i-1)2} - \dot{z}_{(i-1)1}$, $h_{i1} = z_{i1} - \alpha_i$, then Eq. (16) can be rewritten as

$$\begin{aligned}&\theta_{i,m_i}\dot{e}_i \\ &= -\varphi_i^{\mathrm{T}}(\varpi_i)\bar{\theta}_i + x_{i+1} - \Delta_i'(x,t) - \theta_{i,m_i}(z_{(i-1)2} - h_{(i-1)2}) \\ &= x_{i+1} - \Psi_i^{\mathrm{T}}(\varpi_2, z_{(i-1)2})\Theta_i - \Delta_i'(x,t) + \theta_{i,m_i}h_{(i-1)2}\end{aligned} \quad (18)$$

where $\Psi_i(\varpi_2, z_{(i-1)2}) = [\varphi_i^T(\varpi_i), z_{(i-1)2}]^T$, $\Theta_i = [\bar{\theta}_i^{\mathrm{T}}, \theta_{i,m_i}]^{\mathrm{T}}$, and $\Theta_i \in R^{m_i}$.

Then, one can synthesize α_i with the similar structure in Ref. [24] as follows:

$$\alpha_i = \alpha_{ia} + \alpha_{is} \quad (19)$$

where α_{ia} is the adaptive control term and α_{is} is the robust control term. From Eq. (18) and noting the coupled error term between the ith step and the $(i-1)$th step, α_{ia} can be given by

$$\alpha_{ia} = \Psi_i^{\mathrm{T}}(\varpi_i, z_{(i-1)2})\hat{\Theta}_i - e_{i-1} \quad (20)$$

where $\hat{\Theta}_i = [\hat{\bar{\theta}}_i^{\mathrm{T}}, \hat{\theta}_{i,m_i}]^{\mathrm{T}}$.

The robust control term α_{is} contains two parts are

$$\begin{aligned}\alpha_{is} &= \alpha_{is,1} + \alpha_{is,2} \\ \alpha_{is,i} &= -k_{i1}e_i \\ \alpha_{is,2} &= -k_{i2}\tanh(\hbar_ik_{i2}e_i)\end{aligned} \quad (21)$$

where $k_{i1} > 0$, $k_{i2} > 0$, $\hbar_1 > 0$.

Define the following Lyapunov function candidate:

$$V_i = V_{i-1} + V_{e_i} + V_{\tilde{\Theta}_i} + V_{h_{i1}} + V_{h_{(i-1)2}} \quad (22)$$

where $V_{e_i} = \frac{1}{2}\theta_{i,m_i}e_i^2$, $V_{\tilde{\Theta}_i} = \frac{1}{2}\tilde{\Theta}_i^{\mathrm{T}}\Gamma_i^{-1}\tilde{\Theta}_i$, $\tilde{\Theta}_i = \hat{\Theta}_i - \Theta_i$, $V_{h_{i1}} = \frac{1}{2}h_{i1}^2$, and $V_{h_{(i-1)2}} = \frac{1}{2}\theta_{i,m_i}h_{(i-1)2}^2$. The derivative of V_i is $\dot{V}_i = \dot{V}_{i-1} + e_i\theta_{i,m_i}\dot{e}_i + \tilde{\Theta}_i^{\mathrm{T}}\Gamma_i^{-1}\dot{\hat{\Theta}}_i + h_{i1}\dot{h}_{i1} + \theta_{i,m_i}h_{(i-1)2}\dot{h}_{(i-1)2}$.

From Eq. (17), one can get the derivative of $h_{(i-1)2}$ and h_{i1} as follows:

$$\begin{cases} \dot{h}_{(i-1)2} = -\frac{1}{\tau_{(i-1)2}} h_{(i-1)2} \\ \qquad -g_{(i-1)2} \tanh\left(\hat{\lambda}_{(i-1)2} g_{(i-1)2} h_{(i-1)2}\right) - e_i - \ddot{z}_{(i-1)1} \\ \dot{h}_{i1} = -\frac{1}{\tau_{i1}} h_{i1} - g_{i1} \tanh(\hat{\lambda}_{i1} g_{i1} h_{i1}) - e_i - \dot{\alpha}_i \end{cases} \tag{23}$$

From Eqs. (18), (23) and noting $x_{i+1} = \alpha_i + e_{i+1} + h_{i1}$, $\dot{z}_{(i-1)1} = z_{(i-1)2} - h_{(i-1)2}$, it holds that

$$\begin{aligned} \dot{V}_i &= \tilde{\Theta}_i^{\mathrm{T}} \Gamma_i^{-1} \dot{\hat{\Theta}}_i + \dot{V}_{i-1} \\ &\quad + e_i \left(\Psi_i^{\mathrm{T}}(\varpi_2, z_{(i-1)2}) \tilde{\Theta}_i - e_{i-1} + \alpha_{is} + e_{i+1} - \Delta_i'(x,t)\right) \\ &\quad + h_{i1} \left(-\frac{1}{\tau_{i1}} h_{i1} - g_{i1} \tanh(\hat{\lambda}_{i1} h_{i1}) - \dot{\alpha}_i\right) + \theta_{i,m_i} h_{(i-1)2} \\ &\quad \times \left(-\frac{1}{\tau_{(i-1)2}} h_{(i-1)2} - g_{(i-1)2} \tanh\left(\hat{\lambda}_{(i-1)2} h_{(i-1)2}\right) - \ddot{z}_{(i-1)1}\right) \end{aligned} \tag{24}$$

If we choose the adaptive law as $\dot{\hat{\Theta}}_i = -\Gamma_i \Psi_i(\varpi_i, gtz_{(i-1)2}) e_i$ and substitute α_{is} into Eq. (24), then

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} - k_{i1} e_i^2 - \left(k_{i2} e_i \tanh(\hbar_i k_{i2} e_i) + e_i \Delta_i'(x,t)\right) \\ &\quad - e_i e_{i-1} + e_i e_{i+1} - \frac{1}{\tau_{i1}} h_{i1}^2 - (g_{i1} h_{i1} \tanh(\hat{\lambda}_{i1} h_{i1}) + h_{i1} \dot{\alpha}_i) \\ &\quad - \theta_{i,m_i} \left(g_{(i-1)2} h_{(i-1)2} \tanh\left(\hat{\lambda}_{(i-1)2} h_{(i-1)2}\right) + h_{(i-1)2} \right. \\ &\quad \left. \ddot{z}_{(i-1)1}\right) - \frac{\theta_{i,m_i}}{\tau_{(i-1)2}} h_{(i-1)2}^2 \\ &\le \dot{V}_{i-1} - k_{i1} e_i^2 - \left(k_{i2} e_i \tanh(\hbar_i k_{i2} e_i) - |e_i| \cdot \left|\Delta_i'(x,t)\right|\right) \\ &\quad - e_i e_{i-1} + e_i e_{i+1} - \frac{1}{\tau_{i1}} h_{i1}^2 - (g_{i1} h_{i1} \tanh(\hat{\lambda}_{i1} h_{i1}) \\ &\quad - |h_{i1}| \cdot |\dot{\alpha}_i|) - \theta_{i,m_i} \left(g_{(i-1)2} h_{(i-1)2} \tanh(\hat{\lambda}_{(i-1)2} h_{(i-1)2})\right. \\ &\quad \left. - \left|h_{(i-1)2}\right| \cdot \left|\ddot{z}_{(i-1)1}\right|\right) - \frac{\theta_{i,m_i}}{\tau_{(i-1)2}} h_{(i-1)2}^2 \end{aligned} \tag{25}$$

According to Lemma A1, if $k_i 2 > \left|\Delta_i'(x,t)\right|$, $g_{i1} > |\dot{\alpha}_i|_{\max}$, and $g_{(i-1)2} \ge \left|\ddot{z}_{(i-1)1}\right|_{\max}$, we can get $-k_i 2 e_i \tanh(\hbar_i k_i 2 e_i) + |e_i| \cdot |\Delta_i'(x,t)| \le \kappa/\hbar_i$, $-g_{i1} h_{i1} \tanh(\hat{\lambda}_{i1} h_{i1}) + |h_{i1}| \cdot |\dot{\alpha}_i| \le \kappa/\hat{\lambda}_{i1}$, $-g_{(i-1)2} h_{(i-1)2} \tanh\left(\hat{\lambda}_{(i-1)2} h_{(i-1)2}\right) + \left|h_{(i-1)2}\right| \cdot \left|\ddot{z}_{(i-1)1}\right| \le \kappa/\hat{\lambda}_{(i-1)2}$. Then, the derivative of V_i satisfies

$$\begin{aligned} \dot{V}_i &\le -\sum_{j=1}^{i} k_{j1} e_j^2 - \sum_{j=1}^{i} \frac{1}{\tau_{j1}} h_{j1}^2 - \sum_{j=2}^{i} \frac{\theta_{j,m_j}}{\tau_{(j-1)2}} h_{(j-1)2}^2 \\ &\quad + \sum_{j=1}^{i} \varepsilon_j + e_i e_{i+1} \end{aligned} \tag{26}$$

where $\varepsilon_j = \kappa/\hbar_j + \kappa/\hat{\lambda}_j 1 + \kappa/\hat{\lambda}_{(j-1)2}, j = 2, \ldots, i$.

Step n. The error variables related to the design in nth step are $e_n = x_n - z_{(n-1)1}$, the derivative of e_n is $\dot{e}_n = \dot{x}_n - \dot{z}_{(n-1)1}$. With the above defined error variables, a nonsingular fast terminal sliding mode for the nth channel in Eq. (3) can be described as [25]

$$\begin{cases} \sigma_1 = \int_0^t e_n \\ \sigma_2 = \sigma_1 + \frac{\beta}{2-\gamma} |\dot{\sigma}_1 + c\sigma_1|^{2-\gamma} \mathrm{sign}(\dot{\sigma}_1 + c\sigma_1) \end{cases} \tag{27}$$

where $\beta > 0$, $c > 0$, $\gamma = p_1/q_1$, $0 < p_1 < q_1$, p_1, and q_1 are odd integers. The first derivative of σ_2 is [25]

$$\dot{\sigma}_2 = \dot{\sigma}_1 + \beta |\dot{\sigma}_1 + c\sigma_1|^{1-\gamma} (\ddot{\sigma}_1 + c\dot{\sigma}_1) \tag{28}$$

Let

$$z(\sigma_1) = \frac{1}{\beta} |\dot{\sigma}_1 + c\sigma_1|^{\gamma} \mathrm{sign}(\dot{\sigma}_1 + c\sigma_1) + \frac{c}{2-\gamma} (\dot{\sigma}_1 + c\sigma_1) \tag{29}$$

Then, from Eq. (27), we can gain that $\beta |\dot{\sigma}_1 + c\sigma_1|^{1-\gamma} z(\sigma_1) = \dot{\sigma}_1 + c\sigma_2$, and $\dot{\sigma}_2$ can be rewritten as

$$\dot{\sigma}_2 = -c\sigma_2 + \beta''(\ddot{\sigma}_1 + c\dot{\sigma}_1 + z(\sigma_1)) \tag{30}$$

where $\beta'' = \beta |\dot{\sigma}_1 + c\sigma_1|^{1-\gamma}$.

Then, from Eqs. (3), (27), and (30), one can get the following dynamics:

$$\begin{aligned} &\theta_{n,m_n} \dot{\sigma}_2 \\ &= -c\theta_{n,m_n} \sigma_2 + \beta''(\theta_{n,m_n} \ddot{\sigma}_1 + c\theta_{n,m_n} \dot{\sigma}_1 + \theta_{n,m_n} z(\sigma_1)) \\ &= -c\theta_{n,m_n} \sigma_2 + \beta''(\theta_{n,m_n} \dot{x}_n - \theta_{n,m_n} \dot{z}_{(i-1)1} \\ &\quad + c\theta_{n,m_n} \dot{\sigma}_1 + \theta_{n,m_n} z(\sigma_1)) \end{aligned} \tag{31}$$

It is similar to the design in the ith step that a filter is designed to approximate $\dot{z}_{(i-1)1}$ in Eq. (31)

$$\begin{aligned} \dot{z}_{(n-1)2} &= -\frac{1}{\tau_{(n-1)2}} (z_{(n-1)2} - \dot{z}_{(n-1)1}) - \beta'' \sigma_2 \\ &\quad - \left(\frac{1}{\tau_{(n-1)3}} |z_{(n-1)2} - \dot{z}_{(n-1)1}|^r + g_{(n-1)2}\right) \\ &\quad \times \mathrm{sign}(z_{(n-1)2} - \dot{z}_{(n-1)1}) \end{aligned} \tag{32}$$

Define $h_{(n-1)2}$ as $h_{(n-1)2} = z_{(n-1)2} - \dot{z}_{(n-1)1}$, then Eq. (31) can be rewritten as

$$\begin{aligned} \theta_{n,m_n} \dot{\sigma}_2 &= -c\theta_{n,m_n} \sigma_2 + \beta''\left(u - \psi_n^{\mathrm{T}}(\varpi_n) \bar{\theta}_n - \Delta_n'(x,t)\right. \\ &\quad \left. + \theta_{n,m_n} h_{(n-1)2} - \theta_{n,m_n} x_{ned}\right) \\ &= -c\theta_{n,m_n} \sigma_2 + \beta''\left(u - \Psi_n^{\mathrm{T}}(\varpi_n, x_{n,ed}) \Theta_n\right. \\ &\quad \left. -\Delta_n'(x,t) + \theta_{n,m_n} h_{(n-1)2}\right) \end{aligned} \tag{33}$$

where $\Psi_n(\varpi_n, x_{n,ed}) = \left[\varphi_n^{\mathrm{T}}(\varpi_n), x_{ned}\right]^{\mathrm{T}}$, $x_{ned} = z_{(n-1)2} - c\dot{\sigma}_1 - z(\sigma_1)$, $\Theta_n = \left[\bar{\theta}_n^{\mathrm{T}}, \theta_{n,m_n}\right]^{\mathrm{T}}$, and $\Theta_n \in R^{m_n}$.

where u is the control input of the nth channel, and it is also the control input of the whole system as described in Eq. (3), and from Eq. (33), we can gain the control law as follows:

$$u = u_a + u_s \tag{34}$$

where u_a denotes the adaptive control term and u_s is the robust control term. Let

$$u_a = \Psi_n^{\mathrm{T}}(\varpi_n, x_{ned}) \hat{\Theta}_n \tag{35}$$

where $\hat{\Theta}_n = [\hat{\bar{\theta}}_n^{\mathrm{T}}, \hat{\theta}_{n,m_n}]^{\mathrm{T}}$.

While the robust term is given by

$$\begin{cases} u_s = u_{s_1} + u_{s_2} \\ u_{s_1} = -k_{n1} \sigma_2 - k_{n2} |\sigma_2|^r \mathrm{sign}(\sigma_2) \\ u_{s_2} = -k_{n3} \mathrm{sign}(\sigma_2) \end{cases} \tag{36}$$

where $r = p_2/q_2$, $0 < p_2 < q_2$, p_2, and q_2 are odd integers. $k_{n1} > 0$, $k_{n2} > 0$, $k_n 3 > \varsigma_{dn}'$.

Substituting the control effort in Eq. (36) into Eq. (33), one can get that

$$\theta_{n,m_n}\dot{\sigma}_2 = -c\theta_{n,m_n}\sigma_2 + \beta''\big(\Psi_n^{\mathrm{T}}(\varpi_n)\tilde{\Theta}_n + u_s - \Delta'(x,t) + \theta_{n,m_n}h_{(n-1)2}\big) \tag{37}$$

3.2 Adaptive Law Design. In subsection 3.1, the adaptive law is selected as $\dot{\hat{\Theta}}_i = -\Gamma_i\Psi_i(\varpi_i)e_i$ for the ith $(i = 1,\ldots,n-1)$ channel; however, only the tracking error is included in the adaptive law, and the estimated effect of the parameters is not well. In Ref. [16], the composite adaptive law, which includes the tracking error and the prediction error, is used to achieve better parameter estimate. Inspired by this idea, the composite adaptive law is designed for each channel.

For the ith $(i = 1,\ldots,n-1)$ channel in Eq. (3), the system equation can be rewritten as

$$\begin{aligned} x_{i+1} &= \theta_{i,m_i}\dot{x}_i + \varphi_i^{\mathrm{T}}(\varpi_i)\bar{\theta}_i - \Delta_i'(x,t) \\ &= \varphi_i^{\mathrm{T}}(\varpi_i, ; \dot{x}_i)\Theta_i - \Delta_i'(x,t) \end{aligned} \tag{38}$$

where $\varphi_i(\varpi_i,\dot{x}_i) = [\varphi_i^{\mathrm{T}}(\varpi_i),\dot{x}_i]^{\mathrm{T}},\ \ i = 1,\ldots,n-1$. If we define $x_{n+1} = u$, then the system equation in the nth channel can also be rewritten in the form of Eq. (38).

Let x_{i+1} pass through a first-order law filter, then we have

$$x_{(i+1)f} = \phi_{fi}^{\mathrm{T}}(\varpi_i,\dot{x}_i)\Theta_i - f_{\Delta_i'(x,t)} \tag{39}$$

where $\varphi_{fi}(\varpi_i) = [\varphi_{fi}^{\mathrm{T}}(\varpi_i), f_{\dot{x}_i}]^{\mathrm{T}},\ \varphi_{fi}(\varpi_i) = [\varphi_{i1,f}(\varpi_i), \varphi_{i2,f}(\varphi_i),\ldots, \varphi_{i(m_i-1),f}(\varpi_i)]^{\mathrm{T}}$, and

$$\begin{cases} \dot{\varphi}_{i1,f} + \kappa_{fi}\varphi_{i1,f} = \kappa_{fi}\varphi_{i1}(\varpi_i) \\ \vdots \\ \dot{\varphi}_{i(m_i-1),f} + \kappa_{fi}\varphi_{i(m_i-1),f} = \kappa_{fi}\varphi_{i(m_i-1)}(\varpi_i) \\ \dot{x}_{(i+1)f} + \kappa_{fi}x_{(i+1)f} = \kappa_{fi}x_{(i+1)} \\ \dot{f}_{\Delta_i'(x,t)} + \kappa_{fi}f_{\Delta_i'(x,t)} = \Delta_i'(x,t) \end{cases} \tag{40}$$

while $f_{\dot{x}_i}$ is defined as

$$\begin{cases} \dot{x}_{if} + \kappa_{fi}x_{if} = x_i \\ f_{\dot{x}_i} = \kappa_{fi}(x_i - x_{if}) \end{cases} \tag{41}$$

Due to the boundedness of $\Delta_i'(x,t), f_{\Delta_i'(x,t)}$ is bounded. Assume that $f_{\Delta_i'(x,t)} = 0$, then, we can get

$$f_{x_{i+1}} = \phi_{fi}^{\mathrm{T}}(\varpi_i,\dot{x}_i)\Theta_i \tag{42}$$

Define the prediction-error e_{fi} which is similar to that in Ref. [16] as

$$e_{fi} = P_i\hat{\Theta}_i - Q_i \tag{43}$$

While the matrix P_i and Q_i are given by

$$\begin{aligned} P_i &= \textstyle\int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][\phi_f i(\varpi_i)\ell_i(\tau)]^{\mathrm{T}}d\tau \\ Q_i &= \textstyle\int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][x_{(i+1)f}\ell_i(\tau)]d\tau \end{aligned} \tag{44}$$

and $\ell_i(\tau)$ is

$$\ell_i(\tau) = \exp(-\varepsilon_i\tau) \tag{45}$$

where $\kappa_{fi} > 0$, $\varepsilon_i > 0$. κ_{fi} and ε_i are parameters to be designed.

From Eqs. (42)–(45), with some manipulations it holds that [16]

$$\begin{aligned} e_{fi} &= \int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][phi_f i(\phi_i)\ell_i(\tau)]^{\mathrm{T}}d\tau\hat{\Theta}_i \\ &\quad - \int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][x_{(i+1)f}\ell_i(\tau)]d\tau \\ &= \int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][\phi_f i(\varpi_i)\ell_i(\tau)]^{\mathrm{T}}d\tau\hat{\Theta}_i \\ &\quad - \int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][\phi_{fi}^{\mathrm{T}}(\varpi_i,\dot{x}_i)\ell_i(\tau)]d\tau\Theta_i \\ &= \int_0^t [\phi_{fi}(\varpi_i)\ell_i(\tau)][\phi_f i(\varpi_i)\ell_i(\tau)]^{\mathrm{T}}d\tau\tilde{\Theta}_i = P_i\tilde{\Theta}_i \end{aligned} \tag{46}$$

Thus, e_{fi} contains the information of the parameter estimate errors. The adaptive law can be designed as

$$\begin{cases} \dot{\hat{\Theta}}_i = \mathrm{Proj}_{\hat{\Theta}_i}\big(-\Gamma_i\Psi_i(\varpi_i, gtz_{(i-1)2})e_i - \Gamma_i\Lambda_i e_{fi}\big) \\ \qquad i = 1,\ldots,n-1 \\ \dot{\hat{\Theta}}_n = \mathrm{Proj}_{\hat{\Theta}_n}\big(-\Gamma_n\beta''\sigma_2\Psi_n(\varpi_2) - \Gamma_n(\Lambda_n e_{fn} + \Upsilon_n \mathrm{sig}(e_{fn})^r)\big) \end{cases} \tag{47}$$

Remark 3. In Ref. [16], the parameters of the whole nth-order system are estimated uniformly in an adaptive equation, and the dimension of the matrix P is $m\times m$, where $m = \sum_{i=1}^n m_i$. When m is the large, the dimension of the matrix increases awfully. In Eq. (47), the parameters of each channel are estimated by one adaptive equation, respectively. The dimension of matrix P_i is $m_i\times m_i$, and the total dimension of the matrixes is $m_{\mathrm{total}} = \sum_{i=1}^n m_i\times m_i$, which is smaller than $m\times m$. This can reduce the calculation burden.

4 Stability Analysis

Theorem 1. *Suppose that the control law in Eqs.* (34)–(36) *with the adaptive law in Eq.* (47) *is applied to the plant* (3), *and if there exists a time T_0 that $P_i(T_0)(i = 1,\ldots,n)$ is positive definite, then e_n convergences to 0 in finite-time, and e_i converges to a bounded layer exponentially.*

Proof. Define the following Lyapunov function candidate for the nth step:

$$V_\sigma = V_{\sigma_2} + V_{\tilde{\Theta}_n} + V_{h_{(n-1)2}} \tag{48}$$

where $V_{\sigma_2} = \frac{1}{2}\theta_{n,m_n}\sigma_2^2,\ V_{\tilde{\Theta}_n} = \frac{1}{2}\tilde{\Theta}_n^{\mathrm{T}}\Gamma_n^{-1}\tilde{\Theta}_n,\ V_{h_{(n-1)2}} = \frac{1}{2}\theta_{n,m_n}h_{(n-1)2}^2$. The derivative of V_σ is

$$\dot{V}_\sigma = \sigma_2\theta_{n,m_n}\dot{\sigma}_2 + \tilde{\Theta}_n^{\mathrm{T}}\Gamma_n^{-1}\dot{\tilde{\Theta}}_n + \theta_{n,m_n}h_{(n-1)2}\dot{h}_{(n-1)2} \tag{49}$$

From Eq. (32), we can get

$$\begin{aligned} \dot{h}_{(n-1)2} = &-\frac{1}{\tau_{(n-1)2}}h_{(n-1)2} - \frac{1}{\tau_{(n-1)3}}|h_{(n-1)2}|^r\mathrm{sign}(h_{(n-1)2}) \\ &- g_{(n-1)2}\mathrm{sign}(h_{(n-1)2}) - \beta''\sigma_2 - \ddot{z}_{(n-1)1} \end{aligned} \tag{50}$$

Substituting Eqs. (36), (37), (47), and (50) into Eq. (49), one can get

$$\begin{aligned} \dot{V}_\sigma = &-c\theta_{n,m_n}\sigma_2^2 + \beta''\sigma_2\big(u_s - \Delta_n'(x,t) + \theta_{n,m_n}h_{(n-1)2}\big) \\ &+ \beta''\sigma_2\Psi_n^{\mathrm{T}}(\varpi_n)\tilde{\Theta}_n + \tilde{\Theta}_n^{\mathrm{T}}\Gamma_n^{-1}\dot{\tilde{\Theta}}_n - \frac{\theta_{n,m_n}}{\tau_{(n-1)2}}h_{(n-1)2}^2 \\ &- \frac{\theta_{n,m_n}}{\tau_{(n-1)3}}|h_{(n-1)2}|^{1+r} - \beta''\sigma_2\theta_{n,m_n}h_{(n-1)2} \\ &- \big(g_{(n-1)2}|h_{(n-1)2}| + \ddot{z}_{(n-1)1}h_{(n-1)2}\big) \end{aligned} \tag{51}$$

Noting Lemma A2, we have

$$\begin{aligned}&-\beta''\sigma_2\Psi_n^{\mathrm{T}}(\varpi_n)\tilde{\Theta}_n+\tilde{\Theta}_n^{\mathrm{T}}\Gamma_n^{-1}\dot{\tilde{\Theta}}_n\\&\le-\tilde{\Theta}_n^{\mathrm{T}}\Lambda_n e_{f,n}-\tilde{\Theta}_n^{\mathrm{T}}\Upsilon_n\mathrm{sig}(e_{fn})^r\end{aligned}\tag{52}$$

then

$$\begin{aligned}&\dot{V}_\sigma\le\\&-(c\theta_{n,m_n}+k_{n1})\sigma_2^2-k_{n2}|\sigma_2|^{1+r}+\beta''\left(k_{n3}-|\Delta_n'(x,t)|\right)|\sigma_2|\\&-\tilde{\Theta}_n^{\mathrm{T}}\Lambda_n e_{fn}-\tilde{\Theta}_n^{\mathrm{T}}\Upsilon_n\mathrm{sig}(e_{fn})^r-\frac{\theta_{n,m_n}}{\tau_{(n-1)3}}|h_{(n-1)2}|^{1+r}\\&-\frac{\theta_{n,m_n}}{\tau_{(n-1)2}}h_{(n-1)2}^2+\left(-g_{(n-1)2}+|\ddot{z}_{(n-1)1}|\right)|h_{(n-1)2}|\end{aligned}\tag{53}$$

As $k_{n3}>\varsigma_{d_n}'$, $g_{(n-1)2}\ge|\ddot{z}_{(n-1)1}|_{\max}$, it holds that

$$\begin{aligned}&\beta''\left(k_{n3}|\sigma_2|-\Delta_n'(x,t)\sigma_2\right)<0\\&\left(-g_{(n-1)2}+|\ddot{z}_{(n-1)1}|\right)|h_{(n-1)2}|<0\end{aligned}$$

Thus

$$\begin{aligned}\dot{V}_\sigma&\le-(c\theta_{n,m_n}+k_{n1})\sigma_2^2-k_{n2}|\sigma_2|^{1+r}\\&-\tilde{\Theta}_n^{\mathrm{T}}\Lambda_n e_{fn}-\tilde{\Theta}_n^{\mathrm{T}}\Upsilon_n\mathrm{sig}(e_{fn})^r\\&-\frac{\theta_{n,m_n}}{\tau_{(n-1)2}}h_{(n-1)2}^2-\frac{\theta_{n,m_n}}{\tau_{(n-1)3}}|h_{(n-1)2}|^{1+r}\end{aligned}\tag{54}$$

In Eq. (54), the terms related to σ_2 satisfy

$$(c\theta_{n,m_n}+k_{n1})\sigma_2^2+k_{n2}|\sigma_2|^{1+r}=k_{n1}'V_{\sigma_2}+k_{n2}'V_{\sigma_2}^{(r+1)/2}\tag{55}$$

where $k_{n1}'=\frac{2(c\theta_{n,m_n}+k_{n1})}{\theta_{n,m_n}}$, $k_{n2}'=k_{n2}\left(\frac{2}{\theta_{n,m_n}}\right)^{(1+r)/2}$.

While $P_n(T_0)$ is positive definite, the terms related to e_{fn} satisfy

$$-\tilde{\Theta}_n^{\mathrm{T}}\Lambda_n e_{fn}-\tilde{\Theta}_n^{\mathrm{T}}\Upsilon_n\mathrm{sig}(e_{fn})^r\le-\vartheta_{n1}V_{\tilde{\Theta}_n}-\vartheta_{n2}V_{\tilde{\Theta}_n}^{(1+r)/2}\tag{56}$$

where $\vartheta_{n1}=\frac{\lambda_{\min}(\Lambda_n)\chi_1}{\lambda_{\max}(P_n^{\mathrm{T}})}$, $\vartheta_{n2}=\frac{\lambda_{\min}(\Upsilon_3)\chi_1^{(1+r)/2}}{\lambda_{\max}(P_n^{\mathrm{T}})}$, $0<\chi_1\le\frac{2\lambda_{\min}(P_nP_n^{\mathrm{T}})}{\lambda_{\max}(\Gamma_n^{-1})}$.

For the terms related to $h_{(n-1)2}$, it satisfies that

$$\begin{aligned}&-\frac{\theta_{n,m_n}}{\tau_{(n-1)2}}h_{(n-1)2}^2-\frac{\theta_{n,m_n}}{\tau_{(n-1)3}}|h_{(n-1)2}|^{1+r}\\&=-\frac{2}{\tau_{(n-1)2}}V_{h_{(n-1)2}}-\frac{2}{\tau_{(n-1)3}}V_{h_{(n-1)2}}^{(1+r)/2}\end{aligned}\tag{57}$$

As $0<r<1$, $0<(1+r)/2<1$, according to Lemma A3, we can get the following inequality from Eqs. (54)–(57).

$$\dot{V}_\sigma\le-\rho_1V_\sigma-\rho_2V_\sigma^{(1+r)/2}\tag{58}$$

where $\rho_1=\min\{k_{n1}',\vartheta_1,\frac{2}{\tau_{(n-1)2}}\}$, $\rho_2=\min\{k_{n2}',\vartheta_2,\frac{2}{\tau_{(n-1)3}}\}$.

From Eq. (58), we know that V_σ can converge to 0 in finite time. While $V_\sigma=0$, $\sigma_2=0$, and from Eq. (27), we know that e_n can converge to 0 in finite time.

For the ith $(i=1,\ldots,n-1)$ step, the following inequality holds:

$$\begin{aligned}&e_i\Psi_i^{\mathrm{T}}(\varpi_i,z_{(i-1)2})\tilde{\Theta}_i+\tilde{\Theta}_i^{\mathrm{T}}\Gamma_i^{-1}\dot{\tilde{\Theta}}_i\le\\&-\tilde{\Theta}_i^{\mathrm{T}}\Lambda_i e_{fi}=-\tilde{\Theta}_i^{\mathrm{T}}\Lambda_iP_i\tilde{\Theta}_i\le0\end{aligned}\tag{59}$$

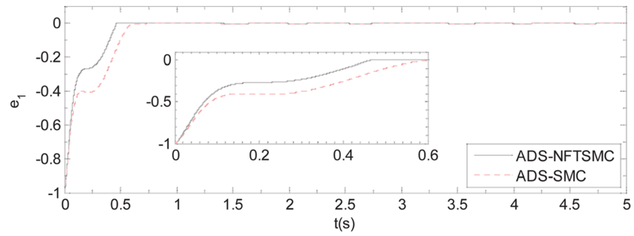

Fig. 1 Error variable e_1

If $P_i(T_0)$ is positive definite, then

$$-\tilde{\Theta}_i^{\mathrm{T}}\Lambda_iP_i\tilde{\Theta}_i\le-\vartheta_i\tilde{\Theta}_i^{\mathrm{T}}\Gamma_i^{-1}\tilde{\Theta}_i\tag{60}$$

where $\vartheta_i=\frac{\lambda_{\min}(\Lambda_i)\chi_i}{\lambda_{\max}(P_i^{\mathrm{T}})}$, $0<\chi_i\le\frac{\lambda_{\min}(P_iP_i^{\mathrm{T}})}{\lambda_{\max}(\Gamma_i^{-1})}$.

Then, from Eq. (24), one can get

$$\begin{aligned}\dot{V}_i&\le-\sum_{j=1}^{i}k_{j1}e_j^2-\sum_{j=1}^{i}\tilde{\Theta}_j^{\mathrm{T}}\Lambda_jP_j\tilde{\Theta}_j-\sum_{j=1}^{i}\frac{1}{\tau_{j1}}h_{j1}^2\\&-\sum_{j=2}^{i}\frac{\theta_{j,m_j}}{\tau_{(j-1)2}}h_{(j-1)2}^2+\sum_{j=1}^{i}\varepsilon_j+e_ie_{i+1}\\&\le-\rho_i'\left\{\sum_{j=1}^{i}\frac{\theta_{j,m_j}}{2}e_j^2+\sum_{j=1}^{i}\frac{1}{2}\tilde{\Theta}_i^{\mathrm{T}}\Gamma_i^{-1}\tilde{\Theta}_i+\sum_{j=1}^{i}\frac{1}{2}h_{j1}^2\right.\\&\left.+\sum_{j=2}^{i}\frac{\theta_{j,m_j}}{2}h_{(j-1)2}^2\right\}+\sum_{j=1}^{i}\varepsilon_j+e_ie_{i+1}\end{aligned}\tag{61}$$

where $\rho_i'=\min\{\rho_i1,\rho_{i2},\rho_{i3},\rho_{i4}\}$, $\rho_{i1}=\min_{j=1,\ldots,i}\{\frac{2k_{j1}}{\theta_{j,m_j}}\}$, $\rho_{i2}=\min_{j=1,\ldots,i}\{2\vartheta_i\}$, $\rho_{i3}=\min_{j=1,\ldots,i}\{\frac{2}{\tau_{j1}}\}$, $\rho_{i4}=\min_{j=1,\ldots,i}\{\frac{2}{\tau_{(j-1)2}}\}$.

From Eq. (22), we know that

$$\begin{aligned}V_i&=\frac{1}{2}\sum_{j=1}^{i}\theta_{j,m_j}e_j^2+\frac{1}{2}\sum_{j=1}^{i}\tilde{\Theta}_j^{\mathrm{T}}\Gamma_j^{-1}\tilde{\Theta}_j\\&+\frac{1}{2}\sum_{j=1}^{i}h_{j1}^2+\frac{1}{2}\sum_{j=2}^{i}\theta_{j,m_j}h_{(j-1)2}^2\end{aligned}\tag{62}$$

So, we can get that

$$\dot{V}_i\le-\rho_i'V_i+\sum_{j=1}^{i}\varepsilon_j+e_ie_{i+1}\tag{63}$$

Define the Lyapunov candidate as

$$V=V_{n-1}+V_\sigma\tag{64}$$

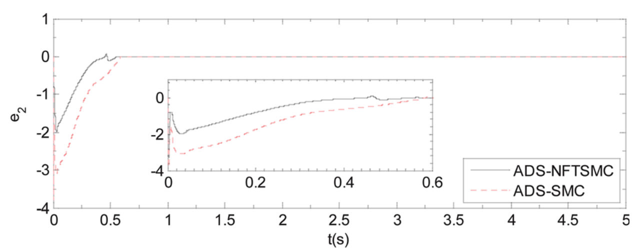

Fig. 2 Error variable e_2

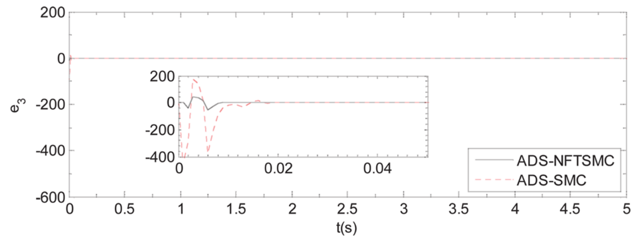

Fig. 3 Error variable e_3

Its derivative is $\dot{V}_n = \dot{V}_{n-1} + \dot{V}_\sigma$. From Eq. (63), we know that $\dot{V}_{n-1} \le -\rho'_{n-1} V_{n-1} + \sum_{j=1}^{n-1} \varepsilon_j + e_{n-1} e_n$, while e_n converges to 0 in finite time, it holds that

$$\begin{aligned}\dot{V}_n &= \dot{V}_{n-1} + \dot{V}_\sigma \\ &\le -\rho'_{n-1} V_{n-1} + \sum_{j=1}^{n-1} \varepsilon_j - \rho_{n1} V_\sigma \\ &\le -\rho'_n V_n + \varepsilon\end{aligned} \tag{65}$$

where $\rho'_n = \min\{\rho'_{n-1}, \rho_n 1\}$, $\varepsilon = \sum_{j=1}^{n-1} \varepsilon_j$.

Remark 4. Due to the use of the nonsingular fast terminal sliding mode control in the nth step, the finite-time convergence of the error e_n can be achieved, and the faster transient response of the system can be obtained.

5 Simulation

In this section, the effectiveness of the proposed method is validated by simulation.

Consider the third order system in semifeedback form

$$\begin{cases}\theta_{1,2}\dot{x}_1 = x_2 - \varphi_1^{\mathrm{T}}(x_1)\bar{\theta}_1 + \Delta_1(x,t) \\ \theta_{2,3}\dot{x}_2 = x_3 - \varphi_2^{\mathrm{T}}(x_1,x_2)\bar{\theta}_2 + \Delta_2(x,t) \\ \theta_{3,4}\dot{x}_3 = u - \varphi_3^{\mathrm{T}}(x_1,x_2,x_3)\bar{\theta}_3 + \Delta_3(x,t)\end{cases} \tag{66}$$

where

$$\begin{aligned}\varphi_1(x_1) &= -x_1 \sin x_1 \\ \varphi_2(x_1,x_2) &= [-x_1 \sin x_2, -x_2 \sin x_1]^{\mathrm{T}} \\ \varphi_3(x_1,x_2,x_3) &= [-x_1 x_3, x_2 \cos x_3, x_3 \sin x_2]^{\mathrm{T}}\end{aligned}$$

and

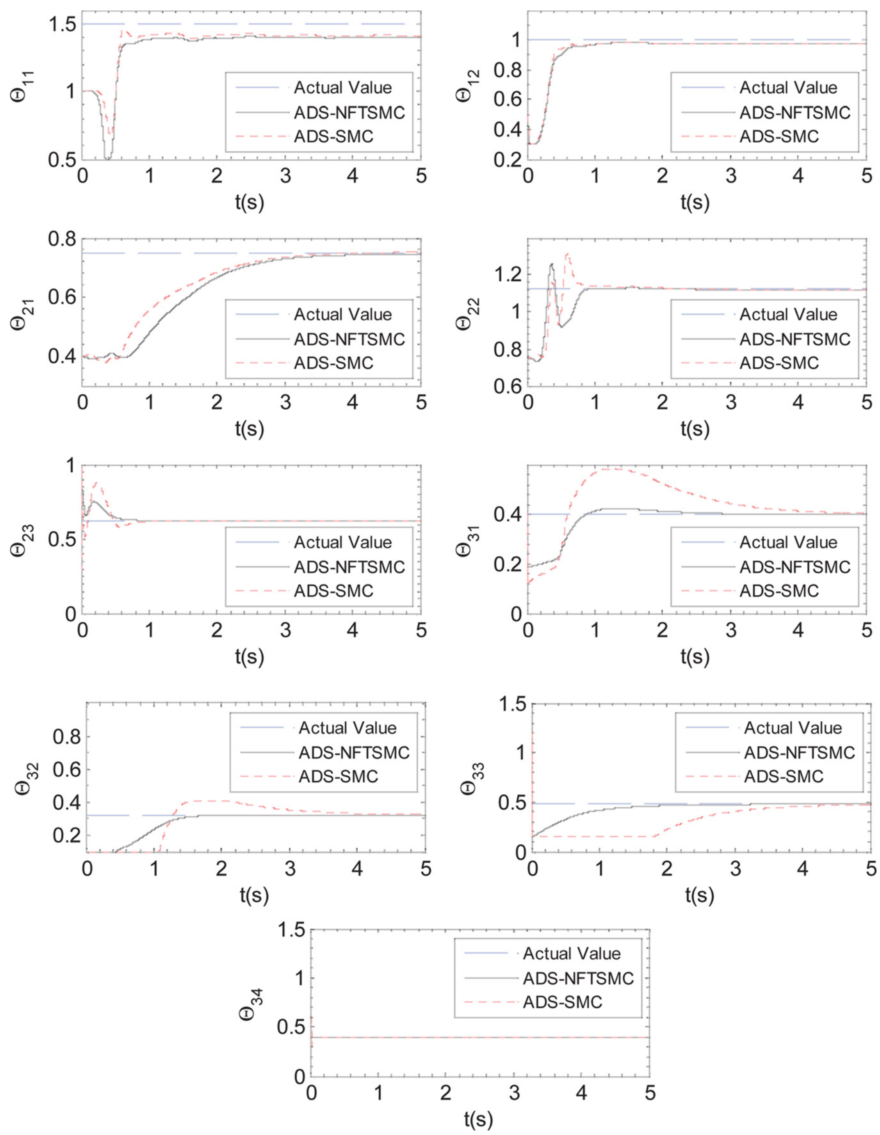

Fig. 4 Estimates of parameters

$$\Delta_1(x,t) = 02x_1^{1/3}\sin x_2$$
$$\Delta_2(x,t) = 032x_1x_2^{1/3}\cos x_3$$
$$\Delta_3(x,t) = 05\sin(10\pi t)$$

The parameters are defined as $\Theta_1 = [\bar{\theta}_1, \theta_{1,2}]^{\mathrm{T}}$, $\Theta_2 = [\bar{\theta}_2^{\mathrm{T}}, \theta_{2,3}]^{\mathrm{T}}$, $\Theta_3 = [\bar{\theta}_3^{\mathrm{T}}, \theta_{3,4}]^{\mathrm{T}}$. The nominal values of the parameters are as follows $\Theta_{1\mathrm{nom}} = [1.5, 1]^{\mathrm{T}}$, $\Theta_{2\mathrm{nom}} = [0.75, 1.125, 0.625]^{\mathrm{T}}$, $\Theta_{3\mathrm{nom}} = [0.4,\ 0.32,\ 0.48,\ 0.4]^{\mathrm{T}}$. The minimal values of the parameters are given by $\Theta_{1\min} = [0.5,\ 0.3]^{\mathrm{T}}$, $\Theta_{2\min} = [0.2,\ 0.4,\ 0.2]^{\mathrm{T}}$, $\Theta_{3\min} = [0.12,\ 0.1,\ 0.15,\ 0.1]^{\mathrm{T}}$. $\Theta_{1\max} = [2.5,\ 2]^{\mathrm{T}}$, $\Theta_{2\max} = [1.5,\ 2.4,\ 1.3]^{\mathrm{T}}$, $\Theta_{3\max} = [1,\ 0.8,\ 1.2,\ 1.5]^{\mathrm{T}}$. The initial parameter values are set as $\hat{\Theta}_1(0) = [1, 0.5]^{\mathrm{T}}$, $\hat{\Theta}_2(0) = [0.4, 0.75, 0.3]^{\mathrm{T}}$, $\hat{\Theta}_3(0) = [0.2, 0.15, 0.2, 0.25]^{\mathrm{T}}$, which are different from the nominal values to test the effect of parametric uncertainties. The objective is to design a control law u such that the output of the closed-loop system can track the desired trajectory $x_d = \sin(\pi t)$ as closely as possible.

Two control methods, i.e., the proposed ADS-NFTSMC and ADS-SMC are implemented. The following values have been chosen for the parameters of ADS-NFTSMC:

Step 1. $k_{11} = 5$, $k_{12} = 0.5$, $\hbar_1 = 90$, $\tau_{11} = \tau_{12} = 0.001$, $g_{11} = 10$, $g_{12} = 20$, $\hat{\lambda}_{11} = \hat{\lambda}_{12} = 100$, $\varepsilon_1 = 0.25$, $\kappa_{f1} = 100$, $\Gamma_1 = [200,\ 10]^{\mathrm{T}}$, $\Lambda_1 = [4,\ 1]^{\mathrm{T}}$.

Step 2. $k_{21} = 5$, $k_{22} = 0.75$, $\hbar_2 = 200$, $\tau_{21} = \tau_{22} = 0.001$, $g_{21} = 25$, $g_{22} = 30$, $\hat{\lambda}_{21} = \hat{\lambda}_{22} = 200$, $\varepsilon_2 = 0.25$, $\kappa_{f2} = 100$, $\Gamma_2 = [10,\ 100,\ 0.1]^{\mathrm{T}}$, $\Lambda_2 = [1,\ 1.2,\ 0.1]^{\mathrm{T}}$.

Step 3. $\beta = 0.1$, $c = 10$, $\gamma = r = 13/15$, $k_{31} = 20$, $k_{32} = 20$, $k_{33} = 1$, $\hbar_3 = 400$, $\Gamma_3 = [1,\ 30,\ 0.2,\ 0.005]^{\mathrm{T}}$, $\Lambda_3 = \Upsilon_3 = [0.15, 0.15, 0.1, 0.01]^{\mathrm{T}}$, $\varepsilon_3 = 0.25$, $\kappa_{f3} = 100$.

For ADS-SMC, if we let $z(\sigma_1) = 0$ and $\sigma_2 = e_n$ in ADS-NFTSMC, then the controller of ADS-SMC can be gain, and the parameters are the same as that of ADS-NFTSMC.

As we can see in Figs. 1–3, the errors converge to zeros gradually in both methods, but the convergent rate of ADS-NFTSMC is much faster than that of ADS-SMC, which indicates that the proposed ADS-NFTSMC improves the tracking performance. Figure 4 shows the composite adaptive law can lead to not only accurate parameter estimates but also fast parameter convergence rate.

6 Discussions

This paper discusses an ADS-NFTSMC approach for uncertain nonlinear systems in semistrict feedback form. The explosion of terms problem is overcome via introducing the DSC technique, and the system performance is enhanced by introducing the NFTSMC technology in the last step. The parameter estimates can converge to their true values faster via designing composite update laws, which leads to a better tracking performance. Finally, it has been shown by simulation that the proposed control method performs reasonably well and that the tracking control objective is achieved.

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Appendix: Preliminaries

Lemma A1. *For $v \neq 0$, $v \in R$, and $\varepsilon > 0$, it holds that* [26]

$$0 < |v| - v\tanh(\varepsilon v) \leq \kappa/\varepsilon \qquad \text{(A1)}$$

where $\kappa = 0.2785$.

Lemma A2. *Assume $a_1 > 0$, $a_2 > 0$, and $0 < b < 1$, then the following inequality holds* [27]:

$$(a_1 + a_2)^b \leqslant a_1^b + a_2^b \qquad \text{(A2)}$$

Lemma A3. *Assume that a continuous positive-definite function $V(t)$ satisfies the following differential inequality* [28]:

$$\dot{V}(t) \leqslant -\alpha V^{\mu}, \forall t \geqslant t_0, V(t_0) \geqslant 0 \qquad \text{(A3)}$$

where $\alpha > 0$, $0 < \mu < 1$ are constants. Then, $V(t) = 0$, $\forall t \geqslant t_1$ with t_1 given by

$$t_1 = t_0 + \frac{V^{1-\mu}(t_0)}{\alpha(1-\mu)} \qquad \text{(A4)}$$

References

[1] Utkin, V., Guldncr, J., and Shi, J., 1999, *Sliding Modes in Electromechanical System*, Taylor & Francis, London.
[2] Kristic, M., Kanellakopoulos, I., and Kokotovic, P., 1995, *Nonlinear and Adaptive Control Design*, Wiley, New York.
[3] Yao, B., and Tomizuka, M., 1994, "Smooth Adaptive Sliding Mode Control of Robot Manipulators With Guaranteed Transient Performance," ASME J. Dyn. Syst. Man Cybern., **SMC-8**, pp. 101–109.
[4] Rios-Bolivar, M., and Zinober, A. S. I., 1997, "Dynamical Adaptive Sliding Mode Output Tracking Control of a Class of Nonlinear Systems," Int. J. Robust Nonlinear Control, **7**, pp. 387–405.
[5] Rios-Bolivar, M., Zinober, A. S. I., and Sira-Ramirez, H., 1996, "Dynamical Sliding Mode Control via Adaptive Input-Output Linearization: A Backstepping Approach," *Robust Control via Variable Structure and Lyapunov Techniques*, F. Garofalo and L. Glielmo, eds., Springer-Verlag, Heidelberg, pp. 15–35.
[6] Koshkouei, A. J., and Zinober, A. S. I., 1999, "Adaptive Sliding Backstepping Control of Nonlinear Semi-Strict Feedback Form Systems," Proceedings of the 7th IEEE Mediterranean Control Conference, Haifa, pp. 2376–2383.
[7] Rios-Bolivar, M., and Zinober, A. S. I., 1998, "A Symbolic Computation Toolbox for the Design of Dynamical Adaptive Nonlinear Control," Appl. Math. Comput. Sci., **8**, pp. 73–88.
[8] Lin, F. J., Shen, P. H., and Hsu, S. P., 2002, "Adaptive Backstepping Sliding Mode Control for Linear Induction Motor Drive," IEE Proc. Electr. Power Appl., **149**(3), pp. 184–194.
[9] Lin, F. J., Chang, C. K., and Huang, P. K., 2007, "FPGA-Based Adaptive Backstepping Sliding-Mode Control for Linear Induction Motor Drive," IEEE Trans. Power Electron., **22**(4), pp. 1222–1231.
[10] Yip, P. P., and Hedrick, J. K., 1998, "Adaptive Dynamic Surface Control: A Simplified Algorithm for Adaptive Backstepping Control of Nonlinear Systems. Int. J. Control, **71**, pp. 959–979.
[11] Hedrick, J. K., and Yip, P. P., 2000, "Multiple Sliding Surface Control: Theory and Application," ASME J. Dyn. Syst., Meas., Control, **122**, pp. 586–593.
[12] Swaroop, D., Hedrick, J. K., Yip, P. P., and Gerdes, J. C., 2000, "Dynamic Surface Control for a Class of Nonlinear Systems," IEEE Trans. Autom. Control, **45**, pp. 1893–1899.
[13] Song, B., Hedrick, J. K., and Howell, A., 2002, "Robust Stabilization and Ultimate Boundedness of Dynamic Surface Control Systems via Convex Optimization," Int. J. Control, **75**, pp. 870–881.
[14] Wang, D., and Huang, J., 2005, "Neural Network-Based Adaptive Dynamic Surface Control for a Class of Uncertain Nonlinear Systems in Strict-Feedback Form," IEEE Trans. Neural Networks, **16**, pp. 195–202.
[15] Yang, Z. J., Toshimasa, N., Shunshoku, K., and Kiyoshi, W., 2007, "Dynamic Surface Control Approach to Adaptive Robust Control of Nonlinear Systems in Semi-Strict Feedback Form," Int. J. Sys. Sci., **38**(9), pp. 709–724.
[16] Chen, J., Li, Z. P., Zhang, G. Z., and Gan, M. G., 2010, "Adaptive Robust Dynamic Surface Control With Composite Adaption Law," Int. J. Adapt. Control Signal Process., **24**(12), pp. 1036–1050.
[17] Lu, X. Y., and Hedrick, J. K., 2000, "Integral Filters From a New Viewpoint and Their Application in Nonlinear Control Design," Proceedings of IEEE International Conference on Contro1 Applications, Anchorage, Alaska, pp. 501–506.
[18] Zhang, H. P., 2004, "Robust Backstepping Control Method Using Sliding Modes With Its Application to the Fin Stabilizer," Dissertation for Doctor Degree, Harbing Engineering University, Harbing.
[19] Wu, Y. X., 2006, "Sliding Mode Control Theory and Its Application in Mobile Manipulators," Dissertation for Doctor Degree, South China University of Technology, Guangzhou.
[20] Li, C. Y., Jing, W. X., and Gao, C. S., 2009, "Adaptive Backstepping-Based Flight Control System Using Integral Filters," Aerosp. Sci. Technol., **13**(2–3), pp. 105–113.

[21] Feng, Y., Yu, X., and Man, Z., 2002, "Non-Singular Adaptive Terminal Sliding Mode Control of Rigid Manipulators," Automatica, **38**(11), pp. 2159–2167.
[22] Zheng, J. F., Feng, Y., Zheng, X. M., Yang, X. Q., 2009, "Adaptive Backstepping-Based Terminal-Sliding Mode Control for Uncertain Nonlinear Systems," Control Theory Appl., **26**(4), pp. 410–414.
[23] Zhou, L., Jiang, C. S., and Du, Y. L., 2009, "A Robust and Adaptive Terminal Sliding Mode Control Based on Backstepping," Control Theory Appl., **26**(6), pp. 678–682.
[24] Wu, J. H., Pu, D. L., and Ding, H., 2007, "Adaptive Robust Motion Control of SISO Nonlinear Systems With Implementation on Linear Motors," Mechatronics, **17**(4–5), pp. 263–270.
[25] Yu, S., Du, J., Yu, X., and Xu, H., 2008, "A Novel Recursive Terminal Sliding Mode With Finite-Time Convergence," Proceedings of the 17th World Congress, the International Federation of Automatic Control, pp. 5945–5949.
[26] Polycarpou, M. M., and Ioannou, P. A., 1993, "A Robust Adaptive Nonlinear Control Design," Proceedings of American Control Conference, *San Francisco*, CA, pp. 1365–1369.
[27] Mitrinovic, D. S., 1970, *Analytic Inequalities*, Springer, New York.
[28] Barambones, O., and Etxebarria, V., 2001, "Robust Sliding Composite Adaptive Control for Mechanical Manipulators With Finite Error Convergence Time," Int. J. Syst. Sci., **32**(4), pp. 1101–1108.