Front. Comput. Sci.
DOI 10.1007/s11704-016-6120-3

REVIEW ARTICLE

A survey on Lyapunov-based methods for stability of linear time-delay systems

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Abstract Recently, stability analysis of time-delay systems has received much attention. Rich results have been obtained on this topic using various approaches and techniques. Most of those results are based on Lyapunov stability theories. The purpose of this article is to give a broad overview of stability of linear time-delay systems with emphasis on the more recent progress. Methods and techniques for the choice of an appropriate Lyapunov functional and the estimation of the derivative of the Lyapunov functional are reported in this article, and special attention is paid to reduce the conservatism of stability conditions using as few as possible decision variables. Several future research directions on this topic are also discussed.

Keywords time-delay system, delay-independent stability, delay-dependent stability, linear matrix inequality, Lyapunov-Krasovskii functional

1 Introduction

Time-delay systems as a kind of infinite-dimensional system are also called hereditary systems, systems with aftereffects, systems with time-lags, systems with dead-time, equations with deviating argument or differential-difference equations. Time-delay is often encountered in many practical engineering systems such as process control systems, manufacturing process, fluid transmissions and networked control systems [1–6]. Generally speaking, time-delay is usually a source of poor performance and instability of a control system. Besides some negative effects, time-delay can also bring some positive effects. It has been shown that the presence of time-delay is helpful for the stabilization of some systems [7–9]. Therefore, stability analysis of time-delay systems is of both practical and theoretical importance.

Studies on stability of time-delay systems date back to the 18th century. Some famous mathematicians such as Euler and Bernoulli did some pioneering works. Systematical studies on this topic began in the 1940s. In recent 15 years, time-delay systems have being in the golden age, and very extensive results have been obtained [10–47]. Most of those results are on the basis of Lyapunov stability theories.

Stability conditions for time-delay systems can be classified into two categories. One is delay-independent stability conditions and the other is delay-dependent stability conditions. Generally speaking, delay-dependent stability conditions are less conservative than delay-independent ones especially when the time-delay is small. Therefore, much attention has been paid to the study of delay-dependent stability conditions. A great number of efforts have been paid to derive a less conservative delay-dependent stability condition. Most of the existing delay-dependent stability conditions are often described in terms of linear matrix inequalities (LMIs) which can be efficiently solved by some numerical algorithms [48].

In this article, an overview of stability of linear time-delay systems is given. Some methods and techniques used to de-

Received February 29, 2016; accepted May 26, 2016

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rive stability conditions for time-delay systems are reviewed. Several future research directions on this topic are also discussed. This article is organized as follows. In Section 2, Lyapunv stability theories for time-delay systems are presented. In Section 3, two important issues for stability analysis of time-delay systems are mainly reviewed. One is how to construct an appropriate Lyapunov functional, and the other is how to estimate the derivative of the Lyapunov functional. In Section 4, several future research directions on this topic are discussed and some conclusions are drawn.

Notation Throughout this article, the superscripts "–1" and "T" stand for the inverse and transpose of a matrix, respectively; $\mathbb{R}^n$ denotes an n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $P > 0$ means that the matrix P is symmetric positive definite; I is an appropriately dimensional identity matrix; some elements in a symmetric matrix are denoted by $*$; $\mathbb{R}^+$ denotes the set of non-negative real numbers; $C_n = C([-h,\ 0], \mathbb{R}^n)$ denotes the Bannch space of continuous functions $\phi : [-h,\ 0] \to \mathbb{R}^n$; $|\cdot|$ denotes the Euclidean norm; for any $\phi \in C_n$, $\|\phi\|_c = \sup\limits_{-h\leqslant s\leqslant 0} |\phi(s)|$ denotes its norm.

2 Lyapunov stability theories for time-delay systems

As for stability of time-delay systems, two Lyapunov methods are often used. One is Lyapunv-Krasovskii functional method, and the other is Lyapunov-Razumikhin function method. In this section, these two methods are reviewed.

Consider the following time-delay system described by

$$\dot{x}(t) = f(t, x_t), \quad t \geqslant t_0, \tag{1}$$

where $f : \mathbb{R} \times C_n \to \mathbb{R}^n$ is continuous and is Lipschitzian in x_t, and $f(t, 0) = 0$, and $x_t = x(t + \theta)$, $-h \leqslant \theta \leqslant 0$.

Before moving on, the following definition of stability of the system described by Eq. (1) is given.

Definition

- If for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta$ implies $|x(t)| < \epsilon$ for all $t \geqslant t_0$, then the trivial solution of Eq. (1) is stable.
- If the trivial solution of Eq. (1) is stable and if δ can be chosen independently of t_0, then the trivial solution of Eq. (1) is uniformly stable.
- If the trivial solution of Eq. (1) is stable and if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta_a$ implies $\lim\limits_{t\to\infty} x(t) = 0$, then the trivial solution of Eq. (1) is asymptotically stable.
- if the trivial solution of Eq. (1) is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that $\|x_{t_0}\|_c < \delta_a$ implies $|x(t)| < \eta$ for $t \geqslant t_0 + T$ and $t_0 \in \mathbb{R}$, then the trivial solution of Eq. (1) is uniformly asymptotically stable.
- If the trivial solution of Eq. (1) is (uniformly) asymptotically stable and if δ_a can be arbitrarily large, finite number, then the trivial solution of Eq. (1) is globally (uniformly) asymptotically stable.

Next, the Lyapunv-Krasovskii functional method and the Lyapunov-Razumikhin function method are presented. Let $x_t(s, \phi)$ be the solution of Eq. (1) at time t with the initial condition $x_s = \phi$.

Theorem 1 (Lyapunov-Krasovskii stability theorem) [49,50] Suppose that f maps $\mathbb{R}\times$(bounded sets in C_n) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions with $u(0) = v(0) = 0$ and $u(\alpha) > 0, v(\alpha) > 0$, for $\alpha > 0$. If there exists a continuous functional $V : \mathbb{R} \times C_n \to \mathbb{R}$ such that

1) $u(|\phi(0)|) \leqslant V(t, \phi) \leqslant v(\|\phi\|_c)$;
2) $\dot{V}(t, \phi) \leqslant -w(|\phi(0)|)$,

where $\dot{V}(t, \phi) = \lim\limits_{\Delta t\to 0^+} \frac{1}{\Delta t} (V(t + \Delta t, x_{t+\Delta t}(t, \phi)) - V(t, \phi))$, then the trivial solution of Eq. (1) is uniformly stable. If $w(\alpha) > 0$ for $\alpha > 0$, then the trivial solution of Eq. (1) is uniformly asymptotically stable. Additionally, if $\lim\limits_{\alpha\to\infty} u(\alpha) = \infty$, then the trivial solution of Eq. (1) is globally uniformly asymptotically stable.

Remark 1 In some cases, the Lyapunov-Krasovskii functional involving the state derivatives $\dot{x}_t$ are very useful in the derivation of the stability conditions. For such a kind of Lyapunov-Krasovskii functional, the conditions in Theorem 1 should be modified. Please refer to Ref. [51] for details.

The requirement of the state variable $x(t)$ in the interval $[t - h,\ t]$ makes the Lyapunov-Krasovskii theorem difficult to apply. Lyapunov-Razumikhin theorem involving only functions rather than functionals can overcome the difficulty to some extent.

Theorem 2 (Lyapunov-Razumikhin stability theorem) [49, 50] Suppose that f maps $\mathbb{R}\times$(bounded sets in C_n) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions with $u(0) = v(0) = 0$ and

$u(\alpha) > 0, v(\alpha) > 0$, for $\alpha > 0$, and v is strictly increasing. If there exists a continuous functional $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

1) $u(|x|) \leqslant V(t, x) \leqslant v(|x|)$;
2) $\dot{V}(t, x(t)) \leqslant -w(|x(t)|)$, if $V(t+\theta, x(t+\theta)) \leqslant V(t, x(t))$ for $\theta \in [-h, 0]$,

where $\dot{V}(t, x(t)) = \frac{\mathrm{d}}{\mathrm{d}t} V(t, x(t)) = \frac{\partial V(t,x(t))}{\partial t} + \frac{\partial V(t,x(t))}{\partial x} f(t, x_t)$, then the trivial solution of Eq. (1) is uniformly stable. If $w(\alpha) > 0$ for $\alpha > 0$, and there exists a continuous non-decreasing function $p(\alpha) > 0$ for $\alpha > 0$, and the above conditions 2 is strengthened to $\dot{V}(t, x(t)) \leqslant -w(|x(t)|)$ if $V(t+\theta, x(t+\theta)) \leqslant p(V(t, x(t)))$ for $\theta \in [-h, 0]$, then the trivial solution of Eq. (1) is uniformly asymptotically stable. Additionally, if $\lim\limits_{\alpha\to\infty} u(\alpha) = \infty$, then the trivial solution of Eq. (1) is globally uniformly asymptotically stable.

3 Stability of linear time-delay systems

For the sake of simplicity, the following linear system with a single discrete delay is considered

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geqslant t_0, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, A and A_1 are system matrices with appropriate dimensions, $\tau(t)$ is a bounded time-varying delay satisfying

$$0 \leqslant l \leqslant \tau(t) \leqslant h, \tag{3}$$

and

$$d_1 \leqslant \dot{\tau}(t) \leqslant d_2. \tag{4}$$

When the time-delay is constant, the system described by Eq. (2) can be rewritten as

$$\dot{x}(t) = Ax(t) + A_1 x(t - h), \quad t \geqslant t_0. \tag{5}$$

Since Lyapunov-based methods often yield sufficient stability conditions, many efforts have been paid to reduce the conservatism of the stability conditions. When using Lyapunov stability theorems for stability analysis, two issues are very crucial. One is the choice of an appropriate Lyapunov functinal, and the other is estimation of the derivative of the Lyapunov functional. In this article, some existing methods and techniques concerning these two issues are reviewed.

3.1 How to choose an appropriate Lyapunov functional?

Next, some methods of choosing an appropriate Lyapunov functional in the existing literature are briefly reviewed.

3.1.1 Some standard Lyapunov functionals

Inspired by the Lyapunov function for a linear system without delay, $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$ is used to derive a stability condition for system described by Eqs. (2)–(4). According to the Lyapunov-Razumikhin theorem, the following stability condition can be obtained.

Theorem 3 The time-delay system described by Eqs. (2)–(4) is asymptotically stable if there exist matrices $P > 0$ and a scalar $q > 0$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + qP & PA_1 \\ * & -qP \end{bmatrix} < 0. \tag{6}$$

A simple Lyapunov-Krasovskii functional as the following form is often used to derive a delay-independent stability condition:

$$V(t, x_t) = x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s. \tag{7}$$

Based on Eq. (7), a delay-independent stability condition can be obtained.

Theorem 4 The time-delay system described by Eqs. (2)–(4) is asymptotically stable if there exist matrices $P > 0$ and $Q > 0$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + Q & PA_1 \\ * & -(1-d_2)Q \end{bmatrix} < 0. \tag{8}$$

Remark 2 When the delay is constant, according to Schur Complement lemma, Eq. (8) is equivalent to $A^{\mathrm{T}}P + PA + Q + PA_1Q^{-1}A_1^{\mathrm{T}}P < 0$. It implies that $A^{\mathrm{T}}P + PA + A_1^{\mathrm{T}}P + PA_1 < 0$ which is a necessary and sufficient condition for the stability of system Eq. (5) with the delay being zero.

Theorem 4 is independent of the time-delay and is very conservative especially when the time-delay is small. Less conservative delay-dependent stability conditions are needed. The following Lyapunov functional is often used in the literature to derive delay-dependent results.

$$\begin{aligned} V(t, x_t) = {} & x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s \\ & + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Zx(s)\mathrm{d}s\mathrm{d}\theta. \end{aligned} \tag{9}$$

Based on the above Lyapunov functional, the following delay-dependent stability condition can be obtained using the free-weighting matrices method [52].

Theorem 5 [52] The time-delay system described by Eqs. (2)–(4) is asymptotically stable if there exist matrices $P > 0$, $Q > 0$, $Z > 0$, $\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \geqslant 0$, and any matrices M and N with appropriate dimensions such that

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & hA^{\mathrm{T}}Z \\ * & \Phi_{22} & hA_1^{\mathrm{T}}Z \\ * & * & -hZ \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} X_{11} & X_{12} & M \\ * & X_{22} & N \\ * & * & Z \end{bmatrix} \geqslant 0. \tag{11}$$

where

$$\Phi_{11} = PA + A^{\mathrm{T}}P + M + M^{\mathrm{T}} + Q + hX_{11};$$
$$\Phi_{12} = PA_1 + M + N^{\mathrm{T}} + hX_{12};$$
$$\Phi_{22} = -N - N^{\mathrm{T}} - (1 - d_2)Q + hX_{22}.$$

3.1.2 Augmented Lyapunov functional

Considering that the first term in Eq. (9), $x^{\mathrm{T}}(t)Px(t)$, only involves the state $x(t)$ but not the delayed state, an augmented Lyapunov functional was proposed in Ref. [53] for system described by Eq. (5).

$$\begin{aligned} V(t, x_t) = {}& \zeta^{\mathrm{T}}(t)P\zeta(t) + \int_{t-h}^{t} \varrho^{\mathrm{T}}(s)Q\varrho(s)\mathrm{d}s \\ & + \int_{-h}^{0}\int_{t+\theta}^{t} \varrho^{\mathrm{T}}(s)Z\varrho(s)\mathrm{d}s\mathrm{d}\theta, \end{aligned} \tag{12}$$

where

$$\zeta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t) \;\; x^{\mathrm{T}}(t-h) \;\; \int_{t-h}^{t} x^{\mathrm{T}}(s)\mathrm{d}s\right]; \tag{13}$$
$$\varrho^{\mathrm{T}}(s) = \left[x^{\mathrm{T}}(s) \;\; \dot{x}^{\mathrm{T}}(s)\right]. \tag{14}$$

Remark 3 Compared with the Lyapunov functional Eq. (9), the augmented Lyapunov functional can lead to less conservative results. Additionally, it is also applicable for systems with time-varying delay, which can been seen in Ref. [54] and references therein.

3.1.3 Triple integral Lyapunov functional

From Eqs. (9) and (12), it can be seen that the Lyapunov functional often contains some integral terms such as $\int_{t-h}^{t} x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s$ and some double integral terms such as $\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Zx(s)\mathrm{d}s\mathrm{d}\theta$. A natural question is that if some triple integral terms are introduced in the Lyapunov functional, what results can be obtained? To answer this question, a Lyapunov functional containing a triple integral term was introduced in Refs. [55, 56].

$$\begin{aligned} V(t, x_t) = {}& \zeta^{\mathrm{T}}(t)P\zeta(t) + \int_{t-h}^{t} \varrho^{\mathrm{T}}(s)Q\varrho(s)\mathrm{d}s \\ & + \int_{-h}^{0}\int_{t+\beta}^{t} \varrho^{\mathrm{T}}(s)Z\varrho(s)\mathrm{d}s\mathrm{d}\beta \\ & + \int_{-h}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\beta, \end{aligned} \tag{15}$$

where $\zeta(t)$ and $\varrho(s)$ are defined in Eqs. (13) and (14), respectively.

It is shown by simulation results that the Lyapunov functional containing triple integral terms is quite effective in reduction of the conservatism of the stability conditions. The idea of introducing triple integral terms into the Lyapunov functional is extended to the time-varying interval delay case in Refs. [57–64], and the following Lyapunov functional is constructed.

$$\begin{aligned} V(t, x_t) = {}& \zeta^{\mathrm{T}}(t)P\zeta(t) + \int_{t-l}^{t} x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s \\ & + \int_{t-h}^{t-l} x^{\mathrm{T}}(s)Q_2x(s)\mathrm{d}s + \int_{t-\tau(t)}^{t-l} x^{\mathrm{T}}(s)Q_3x(s)\mathrm{d}s \\ & + \int_{t-l}^{t} \dot{x}^{\mathrm{T}}(s)Q_4\dot{x}(s)\mathrm{d}s + \int_{t-h}^{t-l} \dot{x}^{\mathrm{T}}(s)Q_5\dot{x}(s)\mathrm{d}s \\ & + \int_{-l}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{-h}^{-l}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{-l}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)Z_3x(s)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{-h}^{-l}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)Z_4x(s)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{-l}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta \\ & + \int_{-h}^{-l}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta, \end{aligned} \tag{16}$$

where

$$\zeta(t) = col\{x(t),\ x(t-l), x(t-h),\ \int_{t-l}^{t} x(s)\mathrm{d}s,\ \int_{t-h}^{t-l} x(s)\mathrm{d}s\}.$$

Inspired by the idea of triple integral Lyapunov functionals, some Lyapunov functionals containing quadruple integral terms were introduced in Refs. [65–67].

3.1.4 Lyapunov functionals using the lower bound of the delay

It can be imagined that more information about the delay is used in the Lyapunov functional and less conservative results can be obtained. When the lower bound of the delay l is not "0", introducing l in the Lyapunov functional can yield less conservative results. The following Lyapunov functionals containing the lower bound of the delay is proposed in Ref. [68].

$$\begin{aligned}V(t,x_t) &= x^{\mathrm{T}}(t)Px(t)+\int_{t-l}^{t}x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s\\&\quad+\int_{t-h}^{t}x^{\mathrm{T}}(s)Q_2x(s)\mathrm{d}s+\int_{t-\tau(t)}^{t}x^{\mathrm{T}}(s)Q_3x(s)\mathrm{d}s\\&\quad+\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1x(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-h}^{-l}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_2x(s)\mathrm{d}s\mathrm{d}\theta.\end{aligned}\tag{17}$$

Remark 4 Besides introducing the information about the lower bound of the delay, another purpose of constructing a Lyapunov functional of the form Eq. (17) is to facilitate considering some useful terms that was ignored in the previous publications.

However, Lyapunov functional Eq. (17) does not use the information about the lower bound of the delay sufficiently. An improved Lyapunov functional using more information about the lower bound of the delay is proposed in Ref. [62]:

$$\begin{aligned}V(t,x_t) &= x^{\mathrm{T}}(t)Px(t)+\int_{t-l}^{t}x^{\mathrm{T}}(s)Q_1x(s)\mathrm{d}s\\&\quad+\int_{t-h}^{t-l}x^{\mathrm{T}}(s)Q_2x(s)\mathrm{d}s+\int_{t-\tau(t)}^{t-l}x^{\mathrm{T}}(s)Q_3x(s)\mathrm{d}s\\&\quad+\int_{-l}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-h}^{-l}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s\mathrm{d}\theta.\end{aligned}\tag{18}$$

In [62], it has been theoretically proved that Lyapunov functional Eq. (18) can lead to less conservative results than Lyapunov functional Eq. (17). However, from the Lyaunov-Krasovskii functional Eq. (18), one can see clearly that there is no information about the lower bound of time-varying delay in the inner integral upper limits of the double integral terms. Therefore, the information about the lower bound of delay is still not fully used and thus may lead to conservative results. Considering the above facts, a Lyapunov functional that sufficiently uses the information about the lower bound of delay was proposed in Refs. [69, 70].

$$\begin{aligned}V(t,x_t) &= \rho^{\mathrm{T}}(t)P\rho(t)+\int_{t-d(t)}^{t-l}x^{\mathrm{T}}(s)Sx(s)\mathrm{d}s\\&\quad+\int_{t-l}^{t}\zeta^{\mathrm{T}}(s)Q_1\zeta(s)\mathrm{d}s+\int_{t-h}^{t-l}\zeta^{\mathrm{T}}(s)Q_2\zeta(s)\mathrm{d}s\\&\quad+\int_{-l}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-h}^{-l}\int_{t+\theta}^{t-l}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-l}^{0}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)Z_3x(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-h}^{-l}\int_{t+\theta}^{t-l}x^{\mathrm{T}}(s)Z_4x(s)\mathrm{d}s\mathrm{d}\theta\\&\quad+\int_{-l}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta\\&\quad+\int_{-h}^{-l}\int_{\theta}^{-l}\int_{t+\lambda}^{t-l}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta,\end{aligned}\tag{19}$$

where

$$\rho(t)=col\{x(t),x(t-l),\ x(t-h),\int_{t-l}^{t}x(s)\mathrm{d}s,\ \int_{t-h}^{t-l}x(s)\mathrm{d}s\},$$

$\zeta(s)=col\{x(s),\dot{x}(s)\}$. It is easy to seen that the the inner integral upper limits of the double integral terms and the triple integral terms contains l. It has been theoretically proved that Lyapunov functional Eq. (19) can lead to less conservative results than Lyapunov functional Eq. (18) [69].

3.1.5 Lyapunov functionals with non-positive-definite matrices

In the Lyapunov functional mentioned above, the Lyapunov matrix P is often required to be positive-definite or semi-positive. Such a constraint can be relaxed by some bounding techniques. For example, consider the following augmented Lyapunov functional

$$\begin{aligned}V(t,x_t) &= \zeta^{\mathrm{T}}(t)P\zeta(t)+\int_{t-h}^{t}\varrho^{\mathrm{T}}(s)Q\varrho(s)\mathrm{d}s\\&\quad+\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,\end{aligned}\tag{20}$$

where $\zeta(t)$ and $\varrho(s)$ are defined in Eqs. (13) and (14), respectively.

If $Q>0$ and $Z>0$, using the inequalities (33)–(34), one can obtain that

$$\int_{t-h}^{t}\varrho^{\mathrm{T}}(s)Q\varrho(s)\mathrm{d}s\geqslant\frac{1}{h}\zeta^{\mathrm{T}}(t)\Gamma_1^{\mathrm{T}}Q\Gamma_1\zeta(t),\tag{21}$$

$$\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\geqslant\frac{2}{h^2}\zeta^{\mathrm{T}}(t)\Gamma_2^{\mathrm{T}}Z\Gamma_2\zeta(t),\tag{22}$$

where

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & I \\ I & -I & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} hI & 0 & -I \end{bmatrix}.$$

Therefore, if

$$P + \frac{1}{h}\Gamma_1^{\mathrm{T}} Q \Gamma_1 + \frac{2}{h^2}\Gamma_2^{\mathrm{T}} Z \Gamma_2 > 0, \tag{23}$$

then the Lyapunov functional Eq. (20) is positive definite. The positive definiteness of matrix P is not necessary, which may lead to less conservative results [65, 71].

3.1.6 Complete quadratic Lyapunov functional

The above mentioned Lyapunov functionals are all *simple* Lyapunov functionals. A limitation of the simple Lyapunov functional is that it is only applicable to the case that the time-delay system is stable when the delay is zero. However, there exist many systems that are unstable for zero delay but stable for non-zero delay. For stability analysis of such systems, simple Lyapunov functionals are not suitable. One can use the complete quadratic Lyapunov functional to solve this problem. The following complete quadratic Lyapunov functional was proposed in [50]:

$$\begin{aligned} V(t, x_t) = {} & x^{\mathrm{T}}(t)Px(t) + 2x^{\mathrm{T}}(t)\int_{-h}^{0} Q(s)x(t+s)\mathrm{d}s \\ & + \int_{-h}^{0}\int_{-h}^{0} x^{\mathrm{T}}(t+s)R(s,\theta)x(t+\theta)\mathrm{d}s\mathrm{d}\theta \\ & + \int_{-h}^{0} x^{\mathrm{T}}(t+s)Z(s)x(t+s)\mathrm{d}s. \end{aligned} \tag{24}$$

In fact, it is not easy to check the existence of such a complete quadratic functional. In order to solve this problem, a discretization scheme was proposed in Refs. [50, 72–74] and the basic idea of the scheme is to choose Q, R, and Z to be piecewise linear matrix-functions.

There are also some other complete Lyapunov functionals in the literature used for the stability analysis of time-delay systems, see Refs. [75–77] and references therein.

3.2 How to estimate the derivative of the Lyapunov functional?

Once a Lyapunov functional is chosen, the next crucial step of the derivation of a stability condition is to estimate the derivative of the Lyapunov functional as tight as possible. Here, some widely used techniques for estimating the derivative of the Lyapunov functional are briefly reviewed.

In this subsection, we take the Lyapunov functional Eq. (9) as an example to illustrate how the existing methods are used to estimate the derivative of the Lyapunov functional. In addition, it is assumed that $l = 0$ in Eq. (3). Taking the derivative of the Lyapunov functional Eq. (9) along the trajectory of the system described by Eq. (2) yields

$$\begin{aligned} \dot{V}(t, x_t) = {} & 2x^{\mathrm{T}}(t)P\dot{x}(t) + x^{\mathrm{T}}(t)Qx(t) \\ & -(1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))Qx(t-\tau(t)) \\ & +h\dot{x}^{\mathrm{T}}(t)Z\dot{x}(t) - \int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s. \end{aligned} \tag{25}$$

From Eq. (25), it is easy to see that the key problem of estimating $\dot{V}(t, x_t)$ is to bound $-\int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$ as tight as possible. In earlier studies, $-\int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$ is often treated as $-\int_{-\tau(t)}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$. Clearly a useful term $-\int_{-h}^{-\tau(t)} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$ is neglected, which may introduce great conservatism. Therefore, the following equation is often used in the current studies [54, 68, 78, 79].

$$\begin{aligned} -\int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s = {} & -\int_{-\tau(t)}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s \\ & -\int_{-h}^{-\tau(t)} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s. \end{aligned} \tag{26}$$

3.2.1 Model transformations and bounding techniques

There are four kinds of model transformations in the literature, please see Ref. [80] for details. Here, we only consider the third transformation and the fourth transformation. By Newton-Leibniz formula, system Eq. (2) can be transformed to

$$\dot{x}(t) = (A + A_1)x(t) - A_1\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s. \tag{27}$$

Substitute Eq. (27) into Eq. (25), and one can see that there is a cross term $-2x^{\mathrm{T}}(t)PA_1\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s$ in $\dot{V}(t, x_t)$. In order to deal with the cross term, some bounding techniques were proposed such as Park's inequality and Moon et al.'s inequality. Applying some bounding techniques to the cross term can produce a term like $\int_{-\tau(t)}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$. Therefore, $-\int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$ in Eq. (25) can be partially eliminated.

Lemma 1 (Park's inequality) [81] For $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, $Z \in \mathbb{R}^{n_a\times n_a} > 0$ and $M \in \mathbb{R}^{n_a\times n_b}$, the following inequality holds

$$-2a^{\mathrm{T}}b \leqslant \begin{bmatrix} a \\ b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z & ZM \\ M^{\mathrm{T}}Z & (M^{\mathrm{T}}Z+I)Z^{-1}(ZM+I) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Lemma 2 (Moon et al.'s inequality) [82] For $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, $N \in \mathbb{R}^{n_a\times n_b}$, $X \in \mathbb{R}^{n_a\times n_a}$, $Y \in \mathbb{R}^{n_a\times n_b}$, and $Z \in \mathbb{R}^{n_b\times n_b}$, if $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix}^{\mathrm{T}} \geqslant 0$ then the following inequality holds

$$-2a^{\mathrm{T}}Nb \leqslant \begin{bmatrix} a \\ b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X & Y-N \\ Y^{\mathrm{T}}-N^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

As pointed in Refs. [83, 84], the essence of the use of the model transformation and bounding techniques is introduction of slack matrices. However, the slack matrices introduced by the model transformation and bounding techniques are not sufficient, which makes the resulting stability conditions much conservative.

3.2.2 Free-weighting matrices method and free-matrix-based integral inequality

In order to overcome the conservatism introduced by model transformation and bounding techniques, a free-weighting matrices method was proposed in Refs. [83, 85–89].

There are two ways to introduce the free-weighting matrices. One way is from the Newton-Leibniz formula. One can see that the following equation holds

$$2\left[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-\tau(t))N_2\right] \times \left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right] = 0. \quad (28)$$

Add the left sides of the above equation to the the derivative of the Lyapunov functional and use the basic inequality $-2a^{\mathrm{T}}b \leqslant a^{\mathrm{T}}Ra + b^{\mathrm{T}}R^{-1}b$ to bound the cross terms, and a delay-dependent stability condition can be obtained.

The other way is by the system equation. It is clear that the following equation holds

$$2\left[x^{\mathrm{T}}(t)T_1 + x^{\mathrm{T}}(t-\tau(t))T_2 + \dot{x}^{\mathrm{T}}(t)T_3\right] \times [\dot{x}(t) - Ax(t) - A_1x(t-\tau(t))] = 0. \quad (29)$$

In addition, the following equation holds

$$2\left[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-\tau(t))N_2 + \dot{x}^{\mathrm{T}}(t)N_3\right] \times \left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right] = 0. \quad (30)$$

Adding the left sides of the above two equations to the derivative of the Lyapunov functional and reserving the term $\dot{x}(t)$ yield another delay-dependent stability condition.

It can be theoretically proved that these two methods of introducing free-weighting matrices are equivalent to each other [53]. However, the second method has an advantage over the first one. It can make the separation between the Lyapunov matrices and the system matrices, which makes it very suitable for robust stability analysis for time-delay systems with polytopic uncertainties [85].

Inspired by the free-weighting matrices method, some free-matrix-based integral inequalities are proposed to deal with the integral term $-\int_{-h}^{0} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s$ directly.

Lemma 3 [90] For any matrices $Z > 0$, M_1, M_2 and a scalar $h > 0$ such that the following integrations are well defined, then

$$-h\int_{t-h}^{t} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s \leqslant \xi^{\mathrm{T}}(t) \begin{bmatrix} M_1 + M_1^{\mathrm{T}} & -M_1 + M_2^{\mathrm{T}} \\ * & -M_2 - M_2^{\mathrm{T}} \end{bmatrix} \xi(t) + \xi^{\mathrm{T}}(t) \begin{bmatrix} M_1^{\mathrm{T}} \\ M_2^{\mathrm{T}} \end{bmatrix} Z^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix} \xi(t), \quad (31)$$

where $\xi^{\mathrm{T}}(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-h) \end{bmatrix}$.

Lemma 4 [91] For any matrices $Z_1 \geqslant 0$, $Z_3 \geqslant 0$, $R \geqslant 0$, Z_2, N_1, N_2 such that the following integrations are well defined, and such that

$$\begin{bmatrix} Z_1 & Z_2 & N_1 \\ * & Z_3 & N_2 \\ * & * & R \end{bmatrix} \geqslant 0,$$

then

$$-\int_{\alpha}^{\beta} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \leqslant \varpi^{\mathrm{T}}\Omega\varpi, \quad (32)$$

where $\varpi^{\mathrm{T}} = \begin{bmatrix} x^{\mathrm{T}}(\beta) & x^{\mathrm{T}}(\alpha) & \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x^{\mathrm{T}}(s)\mathrm{d}s \end{bmatrix}$, $\Omega = (\beta - \alpha)\left(Z_1 + \frac{1}{3}Z_3\right) + N_1(e_1 - e_2) + (e_1 - e_2)^{\mathrm{T}}N_1^{\mathrm{T}} + N_2(2e_3 - 3_1 - e_2) + (2e_3 - 3_1 - e_2)^{\mathrm{T}}N_2^{\mathrm{T}}$, $e_1 = [I\ \ 0\ \ 0]$, $e_1 = [0\ \ I\ \ 0]$, $e_1 = [0\ \ 0\ \ I]$.

3.2.3 Jensen's inequality and Wirtinger-based integral inequality

The free-weighting matrices method can lead to less conservative stability results, but it introduces some free-weighting matrices, which increases the computational complexity. Jensen's inequality does not introduce any additional matrices and has been widely used in the development of delay-dependent stability conditions.

Lemma 5 [55, 92] For a matrix $Z > 0$ and a scalar $h > 0$ such that the following integrations are well defined, then

$$-\int_{t-h}^{t} \varrho^{\mathrm{T}}(s)Z\varrho(s)\mathrm{d}s \leqslant -\frac{1}{h}\int_{t-h}^{t} \varrho^{\mathrm{T}}(s)\mathrm{d}sZ\int_{t-h}^{t} \varrho(s)\mathrm{d}s, \quad (33)$$

$$-\int_{-h}^{0}\int_{t+\theta}^{t} \varrho^{\mathrm{T}}(s)Z\varrho(s)\mathrm{d}s\mathrm{d}\theta \leqslant -\frac{2}{h^2}\int_{-h}^{0}\int_{t+\theta}^{t} \varrho^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta Z\int_{-\tau}^{0}\int_{t+\theta}^{t} \varrho(s)\mathrm{d}s\mathrm{d}\theta. \quad (34)$$

Inequality (33) is the well-known Jensen's inequality proposed in Ref. [92]. Inequality (34) extends the Jensen's inequality to the double-integral case. The Jensen's inequality can also be extended to higher order integral cases, such as triple-integral case. It has been proved that Jensen's inequality is equivalent to the free-weighting matrices method for the constant delay case using the Projection lemma or the Finsler's lemma [93–96]. For the time-varying delay case, these two methods are not equivalent to each other all the time. A generalized Jensen's inequality was proposed in Ref. [97]

Lemma 6 [97] For matrices $Z > 0$ and M, and a scalar $\gamma > 0$ such that the following integrations are well defined, let

$$\int_0^\gamma \varpi(s)\mathrm{d}s = E\psi,$$

then, the following inequality holds

$$\begin{aligned}&\int_0^\gamma \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s\\ &\geqslant \psi^{\mathrm{T}}\left(E^{\mathrm{T}}M + M^{\mathrm{T}}E - \gamma M^{\mathrm{T}}Z^{-1}M\right)\psi.\end{aligned}\tag{35}$$

Conservatism of the Jensen's inequality has been analyzed in Ref. [98]. Jensen's gap can be made arbitrarily small by using an uniform fragmentation with sufficient large order. In order to reduce the undesirable conservatism in the stability conditions caused by Jensen's inequality, a Wirtinger-based integral inequality was proposed in Refs. [99, 100].

Lemma 7 [99, 100] For a matrix $Z > 0$, and a continuous function ϖ in $[a,\ b] \to \mathbb{R}^n$, the following inequality holds

$$\begin{aligned}&\int_a^b \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s\\ &\geqslant \frac{1}{b-a}\int_a^b \varrho^{\mathrm{T}}(s)\mathrm{d}sZ\int_a^b \varrho(s)\mathrm{d}s + \frac{3}{b-a}\Omega^{\mathrm{T}}Z\Omega,\end{aligned}\tag{36}$$

where $\Omega = \int_a^b \varpi(s)\mathrm{d}s - \frac{2}{b-a}\int_a^b\int_a^\theta \varpi(s)\mathrm{d}s\mathrm{d}\theta$.

Remark 5 It has been theoretically proved in Refs. [101, 102] that the Jensen's inequality (33) is equivalent to inequality (31) and Wirtinger-based integral inequality (36) is equivalent to inequality (32).

The above Wirtinger-based integral inequality was extended to the double-integral case in Ref. [103].

Lemma 8 [103] For a matrix $Z > 0$, and a continuous function ϖ in $[a,\ b] \to \mathbb{R}^n$, the following inequality holds

$$\begin{aligned}&\frac{(b-a)^2}{2}\int_a^b\int_\theta^b \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s\mathrm{d}\theta\\ &\geqslant \int_a^b\int_\theta^b \varpi^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta Z\int_a^b\int_\theta^b \varpi(s)\mathrm{d}s\mathrm{d}\theta + 2\Xi^{\mathrm{T}}Z\Xi,\end{aligned}\tag{37}$$

where $\Xi = -\int_a^b\int_\theta^b \varpi(s)\mathrm{d}s\mathrm{d}\theta + \frac{3}{b-a}\int_a^b\int_\lambda^b\int_\theta^b \varpi(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\lambda$.

Furthermore, the Wirtinger-based integral inequality was refined as the following one in Ref. [104].

Lemma 9 [104] For a matrix $Z > 0$, and a continuous function ϖ in $[a,\ b] \to \mathbb{R}^n$, the following inequality holds

$$\begin{aligned}&\int_a^b \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s\\ &\geqslant \frac{1}{b-a}\int_a^b \varrho^{\mathrm{T}}(s)\mathrm{d}sZ\int_a^b \varrho(s)\mathrm{d}s\\ &\quad+\frac{3}{b-a}\Omega^{\mathrm{T}}Z\Omega + \frac{5}{b-a}\Upsilon^{\mathrm{T}}Z\Upsilon,\end{aligned}\tag{38}$$

where Ω is as defined in Lemma 7, and $\Upsilon = \int_a^b \varpi(s)\mathrm{d}s - \frac{6}{b-a}\int_a^b\int_a^\theta \varpi(s)\mathrm{d}s\mathrm{d}\theta + \frac{12}{(b-a)^2}\int_a^b\int_a^\lambda\int_a^\theta \varpi(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\lambda$.

Clearly, Lemma 9 is less conservative than Lemma 7 since an additional positive term $\frac{5}{b-a}\Upsilon^{\mathrm{T}}Z\Upsilon$ is introduced. A more generalized form for inequality (38) was put forward in Ref. [105] and named "auxiliary function-based integral inequality".

Lemma 10 [105] For s matric $Z > 0$, a continuous function ϖ in $[a,\ b] \to \mathbb{R}^n$, and auxiliary scalar functions $p_1(s)$ and $p_2(s)$, the following inequality holds

$$\begin{aligned}&\int_a^b \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s\\ &\geqslant \frac{1}{b-a}\int_a^b \varrho^{\mathrm{T}}(s)\mathrm{d}sZ\int_a^b \varrho(s)\mathrm{d}s\\ &\quad+\left(\int_a^b p_1^2(s)\mathrm{d}s\right)^{-1}\int_a^b p_1(s)\varpi^{\mathrm{T}}(s)\mathrm{d}sZ\int_a^b p_1(s)\varpi(s)\mathrm{d}s\\ &\quad+\left(\int_a^b p_2^2(s)\mathrm{d}s\right)^{-1}\int_a^b p_2(s)\varpi^{\mathrm{T}}(s)\mathrm{d}sZ\int_a^b p_2(s)\varpi(s)\mathrm{d}s,\end{aligned}\tag{39}$$

where $\int_a^b p_i(s)\mathrm{d}s = 0, \quad i = 1, 2, \quad \int_a^b p_i(s)p_2(s)\mathrm{d}s = 0.$

Remark 6 If choose $p_1(s) = (s-a) - \frac{b-a}{2}$ and $p_2(s) = (s-a)^2 - (b-a)(s-a) + \frac{(b-a)^2}{6}$ in (39), inequality (38) can be obtained. Therefore, Lemma 10 is more general than Lemma 9. An auxiliary function-based double integral inequality can also be seen in [105].

Most recently, the Wirtinger-based integral inequality has been extended a more general one as follows.

Lemma 11 [106] For $m = 1, 2, \ldots$, define

$$\begin{cases} \psi_{2m}(s) = \left(s - \dfrac{a+b}{2}\right)^{2m} + \sum\limits_{i=0}^{m-1} \alpha_{mi} \left(s - \dfrac{a+b}{2}\right)^{2i}; \\ \psi_{2m+1}(s) = \left(s - \dfrac{a+b}{2}\right)^{2m+1} + \sum\limits_{i=0}^{m-1} \beta_{mi} \left(s - \dfrac{a+b}{2}\right)^{2i+1}, \end{cases}$$

assume $\forall i = 0, 1, 2, \ldots, m-1$

$$\int_a^b \psi_{2m}(s)\psi_{2i}(s)\mathrm{d}s = \int_a^b \psi_{2m+1}(s)\psi_{2i+1}(s)\mathrm{d}s = 0, \quad (40)$$

and then the following inequality holds

$$\int_a^b \varpi^{\mathrm{T}}(s)Z\varpi(s)\mathrm{d}s \geqslant \sum_{i=0}^{\infty} \frac{1}{p_i}\Omega_i^{\mathrm{T}}(s)Z\Omega_i, \quad (41)$$

where $p_i = \int_a^b \psi_i^2 \mathrm{d}s > 0$ and $\Omega_i(s) = \int_a^b \psi_i(s)\varpi(s)\mathrm{d}s$.

3.2.4 Convex combination method

As seen in Eq. (26), when estimate the derivative of the Lyapunov functional, $-\int_{-h}^0 \dot{x}^{\mathrm{T}}(s)Zx(s)\mathrm{d}s$ is often divided into two terms to obtain a less conservative result. Such a manipulation usually results in a stability condition being of the form $\Phi + \tau(t)X_1 + (h-\tau(t))X_2 < 0$. In the earlier studies, it was often treated as $\Phi + hX_1 + hX_2 < 0$, which may introduce much conservatism. In Ref. [107], a convex combination method was proposed to deal with such a situation. $\tau(t)X_1 + (h-\tau(t))X_2$ can be seen as a convex combination of X_1 and X_2. Therefore, $\Phi + \tau(t)X_1 + (h-\tau(t))X_2 < 0$ is equivalent to $\Phi + hX_1 < 0$ and $\Phi + hX_2 < 0$ considering $0 \leqslant \tau(t) \leqslant h$.

The convex combination method was extended to the second order case in Ref. [108].

Lemma 12 [108] For symmetric matrices X_0, X_1, and $X_2 \geqslant 0$, let $f(\alpha) = X_0 + \alpha X_1 + \alpha^2 X_2$, then

$$f(\alpha_1) < 0 \ \text{ and } \ f(\alpha_2) < 0 \Rightarrow f(\alpha) < 0, \quad \forall \alpha \in [\alpha_1, \ \alpha_2].$$

When using Jensen's inequality or its generalized version to deal with the derivative of the Lyapunov functional, the time-varying coefficients $\frac{1}{\tau(t)}$ and $\frac{1}{h-\tau(t)}$ will be produced. For example,

$$-\int_{t-\tau(t)}^t \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s \leqslant -\frac{1}{\tau(t)}\int_{t-\tau(t)}^t \dot{x}^{\mathrm{T}}(s)\mathrm{d}s\, Z \int_{t-\tau(t)}^t \dot{x}(s)\mathrm{d}s, \quad (42)$$

$$-\int_{t-h}^{t-\tau(t)} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s \leqslant -\frac{1}{h-\tau(t)}\int_{t-h}^{t-\tau(t)} \dot{x}^{\mathrm{T}}(s)\mathrm{d}s\, Z \int_{t-h}^{t-\tau(t)} \dot{x}(s)\mathrm{d}s. \quad (43)$$

In order to deal with these time-varying coefficients with less conservatism, the following reciprocally convex approach was introduced in Ref. [109].

Lemma 13 For matrices $X_1 \geqslant 0, X_2 \geqslant 0, \ldots, X_n \geqslant 0$, vectors $\xi_1, \xi_2, \ldots, \xi_n$ with appropriate dimensions, positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_i \alpha_i = 1$ and any matrices R_{ij} ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, i-1$) such that

$$\begin{bmatrix} X_i & R_{ij} \\ * & X_j \end{bmatrix} \geqslant 0,$$

then, the following inequality holds

$$\sum_{i=1}^n \frac{1}{\alpha_i}\xi_i^{\mathrm{T}} X_i \xi_i \geqslant \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_1 & R_{12} & \cdots & R_{1n} \\ * & X_2 & \cdots & R_{2n} \\ * & * & & \vdots \\ * & * & \cdots & X_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}. \quad (44)$$

The above method was further extended to the second-order reciprocally convex combination method in [110].

Another way to deal with these time-varying coefficients is the joint use of the convex combination method and the delay-partitioning idea [95]. For example, an inequality $\frac{h-l}{\tau(t)-l} + \frac{h-l}{h-\tau(t)} \geqslant \frac{2N}{i-1} + \frac{2N}{2N-i-1}$, $i = 0, 1, \ldots, N-1$ was proposed in [111]. Similar methods can be seen in [112].

4 Conclusions

This article has surveyed some existing methods and techniques for stability analysis of linear time-delay systems. Choosing an appropriate Lyapunov functional and dealing with the derivative of the Lyapunov functional are crucial issues in derivation of a stability condition using Lyapunv stability theories. Some existing methods and techniques for these two issues have been reviewed. Relationship between those methods and techniques are also discussed.

It is clear that the research on time-delay systems has been booming in recent decades. Many important results for the stability of time-delay systems have been obtained. However, there still are some important issues that are not completely solved. We would like to conclude this paper by elaborating some of our opinions on the future research directions.

1) In practice, some systems are unstable when the delay is set to zero. The simple Lyapunov functionals in existing

literature are not applicable to such systems. The complete Lyapunov functional can solve this problem. However, the complete Lyapunov functional suffers from a high computational complexity coming from the estimation of its derivative. Therefore, how to construct a simple Lyapunov functional for zero-delay unstable systems and derive a tractable stability condition is a problem deserving study.

2) In recent several years, some improved version of Jensen's inequalities have been proposed. However, these inequalities do not use the information about the systems model. In another word, these inequalities are not related with the system matrices. If the system dynamics are considered in the derivation of the new integral inequalities, a less conservative result may be obtained.

3) In the existing results, the system is often assumed to be time-invariant. Results for stability of linear time-varying systems with delays are rare compared with the rich results for linear time-invariant time-delay systems. Developing stability conditions for linear time-varying systems with delays is a topic worthy of study.

Acknowledgements This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 61104097, 61321002, 61120106010, 61522303, and U1509215), Program for Changjiang Scholars and Innovative Research Team in University (IRT1208), ChangJiang Scholars Program, Beijing Outstanding PhD Program Mentor (20131000704), Program for New Century Excellent Talents in University (NCET-13-0045), and Beijing Higher Education Young Elite Teacher Project.

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