Event-triggered consensus for linear continuous-time multi-agent systems based on a predictor

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\begin{abstract}
In this paper, the problem of an event-triggered consensus for a linear continuous-time multi-agent system is investigated. A new event-triggered consensus protocol based on a predictor is proposed to achieve consensus while not requiring continuous communication among agents. The predictor utilizes an artificial closed-loop system to predict the future state of each agent. With the proposed consensus protocol, each agent only needs to monitor its own states to determine its event-triggered instants. When an event of an agent is triggered, the agent immediately updates its consensus protocol and sends its state information to its neighbors. When an agent receives state information from its neighbors, the agent immediately updates its consensus protocol and predictor. A necessary and sufficient condition that solves the consensus problem is derived. Moreover, it is proved that Zeno behaviors are excluded. Finally, some numerical examples are given to illustrate that, with the proposed protocol, a multi-agent system can achieve consensus while greatly reducing event-triggered times.
\end{abstract}

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1. Introduction

In the 1970s, the definition of an agent was proposed in the field of intelligence [25]. Since then, increasing numbers of researchers have paid attention to multi-agents, and many results have been obtained. To mention a few, the consensus problem of multi-agent systems with a directed communication topology was investigated in [27], and a theoretical framework for this problem was built. The consensus problem of multi-agent systems with first-order integrator dynamics, active leaders and variable interconnection topologies was considered in [18]. For multi-agent systems with second-order integrator dynamics, a necessary and sufficient condition for the consensus was proposed in [33]. The leader-following consensus problem of second-order nonlinear multi-agent systems with general topologies was studied without assuming that the interaction digraph was strongly connected or contained a directed spanning tree in [31], while the leader-following consensus for a general multi-agent system was addressed in [6]. For multi-agent systems with high-order integrator dynamics, a necessary and sufficient condition was proposed for the consensus problem in [17]. The consensus for high-order linear multi-agent systems with time delays in both the communication channels and control inputs was investigated in [42]. The consensus problem of multi-agent systems with fixed/switching communication topology was investigated in [20], using the
Lyapunov method. The existence of consensus protocols for linear continuous-time/discrete-time multi-agent systems with fixed communication topology was proved in [14,24]. A leader-following consensus problem for multi-agent systems subject to unknown-but-bounded process and measurement noises was developed in [11]. The recent advances in distributed cooperative control under a sampled-data setting were presented in [10]. Some other results concerning multi-agent systems can be seen in [2,12,13,26] and references therein.

It should be noted that, the above results are all based on the assumption of continuous communication between agents to implement the consensus protocol. However, continuous communication between agents is difficult in practice, since the network bandwidth and the energy of agents are limited [13,38]. Moreover, continuous communication between agents wastes communication resources [3,8,19,35]. Among efforts to avoid continuous communication, an event-triggered strategy has received increasing attention [5,39,40]. A consensus protocol was designed in [4] for multi-agent systems with first-order integrator dynamics based on a self-triggered strategy. Event-triggered consensus protocols were designed for multi-agent systems with first-order/second-order integrator dynamics in [30]. A new distributed transmission strategy was proposed in [15], which permits the event-triggering condition to be intermittently examined at constant sampling instants. A dynamic event-triggered communication mechanism was addressed in [9] to obtain better resource efficiency. In [34], two event-triggered consensus protocols were designed for multi-agent systems with general linear dynamics, while both protocols were only effective for the undirected communication topology. The event-triggered consensus problem for multi-agent systems with general linear dynamics and directed communication topology was investigated in [43]. The consensus protocol in [43] could make multi-agent systems achieve consensus without continuous communication, but the state differences between agents would merely converge to the neighborhood of zero. In [32], a decentralized consensus protocol was designed to make the state differences between agents ultimately converge to zero based on an event-triggered strategy, and a necessary and sufficient condition was proposed for the consensus.

In this paper, the problem of an event-triggered consensus protocol for multi-agent systems with general linear continuous-time dynamics under directed graphs is investigated. The main contributions of the paper can be summarized as follows.

1. A new event-triggered consensus protocol based on a predictor is developed. The predictor utilizes an artificial closed-loop system to estimate the state difference between each agent and its neighboring agents. The scheme can make the multi-agent system achieve consensus with fewer event-triggered times than some existing methods.

2. Most existing literature focuses on a first-order integrator network, second-order integrator network, or linear multi-agent system under undirected graphs. However, the event-triggered consensus problem with general linear dynamics under directed graphs is solved in this paper.

3. A necessary and sufficient condition that solves the consensus problem is derived. Zeno behavior is proved to be avoided.

The rest of this paper is organized as follows. Some useful notation and the graph theory are introduced in Section 2. The design of the consensus protocol based on the event-triggered strategy is given in Section 3. In Section 4, the analysis of the consensus protocol is presented. A numerical example is given in Section 5 to illustrate the efficiency and the advantages of the event-triggered consensus protocol presented in this paper. Section 6 provides our conclusions.

2. Notation and graph theory

Standard notations are used throughout this paper. \( R^{m \times n} \) denotes the set of \( m \times n \) real matrices; \( 0_{m \times n} \) indicates an \( m \times n \) zero matrix; \( I_n \) stands for an \( n \times n \) identity matrix; \( 1_n \) represents an \( n \times 1 \) column vector whose entries are all 1; a diagonal matrix with \( x_i(1 \leq i \leq n) \) is denoted by \( \text{diag}(x_1, x_2, \ldots, x_n) \); \( A \otimes B \) means the Kronecker product of \( A \) and \( B \); \( \| \cdot \| \) denotes the Euclidean norm for vectors or the induced 2-norm for matrices; \( \text{Re}(\cdot) \) means the real part of a complex number; and \( \lambda_i(\cdot) \) is the \( i \)th eigenvalue of a matrix.

The communication topology among the \( N \) agents is represented by a weighted graph \( \mathcal{G} = (V, \mathcal{E}, \mathcal{A}) \). \( N \) agents in a multi-agent system are regarded as nodes \( V = 1, 2, \ldots, N \) of the graph \( \mathcal{G} \). A directed graph contains a directed spanning tree if there are directed paths from one node to every other node. The adjacency matrix is defined as \( \mathcal{A} = [a_{ij}] \in R^{N \times N} \) associated with the directed graph \( \mathcal{G} \). Assume that for all \( i \in V \), \( a_{ii} = 0 \), \( a_{ij} > 0 \) if \( e_{ij} \in \mathcal{E} \) and \( a_{ij} = 0 \) otherwise. The directed edge \( e_{ij} \in \mathcal{E} \) indicates that agent \( i \) can receive information from agent \( j \). Therefore, agent \( i \) can be called as agent \( j \)'s out-neighbor agent and agent \( j \) can be called as agent \( i \)'s in-neighbor agent. \( \mathcal{L} = [l_{ij}] \in R^{N \times N} \) denotes the Laplacian matrix of the directed graph \( \mathcal{G} \), where \( l_{ii} = \sum_{j=1}^{N} a_{ij} \) and \( l_{ij} = -a_{ij}(i \neq j) \).

3. Design of the event-triggered consensus protocol

Consider a linear continuous-time multi-agent system consisting of \( N \) agents, where the dynamic of agent \( i \) is described by

\[
\dot{x}_i(t) = Ax_i(t) + Bu_i(t)
\]

where \( x_i(t) \in R^{n \times 1} \) and \( u_i(t) \in R^{m \times 1} \) are the state and the control input, respectively. \( A \in R^{n \times n} \) and \( B \in R^{n \times m} \) are constant matrices. The communication topology among the \( N \) agents can be described by a directed weighted graph \( \mathcal{G} \).

**Assumption 1.** The matrix pair \((A, B)\) in (1) is stabilizable and the graph \( \mathcal{G} \) contains a directed spanning tree.
The well-known consensus protocol for the multi-agent system (1) is

$$u_i(t) = K \sum_{j=1}^{N} a_{ij} \left[ x_i(t) - x_j(t) \right]$$ (2)

To apply protocol (2), a continuous communication between agent $i$ and $j$ is needed. To save communication costs among agents, an event-triggered strategy is applied to design the consensus protocol. Under the event-triggered strategy, an event is designed for each agent in the multi-agent system. An agent broadcasts its current information to its out-neighbor agents only when its event is triggered. The following consensus protocol is designed

$$u_i(t) = K \sum_{j=1}^{N} a_{ij} \left[ \hat{x}_i(t) - \hat{x}_j(t) \right]$$

$$= K \sum_{j=1}^{N} a_{ij} \left[ \left( e^{A(t-t_{ij}^i)} x_i(t_{ij}^i) + \int_{t_{ij}^i}^{t} e^{A(t-s)} B \tilde{u}_i(s) ds \right) - \left( e^{A(t-t_{ij}^i)} x_j(t_{ij}^i) + \int_{t_{ij}^i}^{t} e^{A(t-s)} B \tilde{u}_j(s) ds \right) \right]$$ (3)

where $K \in \mathbb{R}^{m \times n}$ is the feedback gain matrix to be determined. $t_{ij}^i$ is the most recent triggering instant of agent $i$, and $k_i = 1, 2, 3, \ldots$ represents the sequence number of the triggering instant of agent $i$. $x_i(t_{ij}^i)$ is the latest broadcast state of agent $i$. $\hat{x}_i(t)$ and $\hat{u}_i(t)$ respectively represent the estimates of the state and the control input of agent $i$. Then the measurement error of agent $i$ is defined as

$$e_i(t) = e^{A(t-t_{ij}^i)} x_i(t_{ij}^i) + \int_{t_{ij}^i}^{t} e^{A(t-s)} B \tilde{u}_i(s) ds - x_i(t)$$ (4)

and the event function is

$$f_i(t) = \| e_i(t) \| - c_1 e^{-\alpha t}$$ (5)

where $c_1 > 0$, $0 < \alpha < -\text{maxRe}(\lambda_i(\Pi))$, and $\Pi$ is defined after (18).

When the event function $f_i(t) \geq 0$, agent $i$'s event is triggered. Then agent $i$ sends its current information including its state and state differences between agent $i$ and its in-neighbor agents to its out-neighbor agents and updates its consensus protocol. At the same time, the measurement error $e_i(t)$ is reset to zero. If the event function $f_i(t) < 0$, then communication from agent $i$ to its out-neighbor agents is unnecessary until the next event is triggered. However, agent $i$ will update its consensus protocol as soon as it receives information from its in-neighbor agents. Events of all agents are assumed to be triggered at the initial instant.

From (3), it can be seen that the main challenge of the event-triggered consensus protocol proposed in this paper is how to obtain the estimate of the control input. Next, a method of estimating the control input is presented.

Define $\dot{\theta}_{ij}(t) = x_i(t) - x_j(t)$. Then (2) can be rewritten as

$$u_i(t) = K \sum_{j=1}^{N} a_{ij} \dot{\theta}_{ij}(t)$$ (6)

If applying protocol (2) to system (1), it is clear that

$$\dot{\theta}_{ij}(t) = \dot{x}_i(t) - \dot{x}_j(t) = A \theta_{ij}(t) + BK \sum_{p=1}^{N} a_{ij} \theta_{ip}(t) - BK \sum_{q=1}^{N} a_{ij} \theta_{jq}(t)$$

$$= A \theta_{ij}(t) + B \Omega_i(t)$$ (7)

where $\Omega_i(t) = K (\sum_{p=1}^{N} a_{ij} \theta_{ip}(t) - \sum_{q=1}^{N} a_{ij} \theta_{jq}(t))$.

From (6), we know that the estimate problem of agent $i$'s control input can be transformed to the estimate problem of the state differences between agent $i$ and its neighborhood agents. On the basis of (7), the following predictor is designed to estimate this

$$\hat{\theta}_{ij}(t) = e^{A(t-t_{ij}^i)} \theta_{ij}(t_{ij}^i) + \int_{t_{ij}^i}^{t} e^{A(t-s)} B \hat{\Omega}_i(s) ds$$ (8)

where $\hat{\Omega}_i(t) = K (\sum_{p=1}^{N} a_{ij} \theta_{ip}(t_{kj}^p) - \sum_{q=1}^{N} a_{ij} \theta_{jq}(t_{kj}^q))$. and $t_{kj}^i$, $t_{kj}^p$, and $t_{kj}^q$ are the most recent triggering instants of agents $i$, $p$, and $q$, respectively.

**Remark 1.** It should be noted that (8) utilizes the artificial closed-loop system (7) to predict the future state. From the simulation results, we can see that predictor (8) works well.

From (3) and (8), it can be seen that agent $i$'s in-neighbor agent $p$ sends its current state $x_{ip}(t_{kj}^p)$ and state differences between the agent and its neighbor agents to agent $i$ at its triggering instant $t_{kj}^i$. As soon as agent $i$ receives the information
from agent \( p \), agent \( i \) updates the consensus protocol and the state difference between itself and agent \( p \), i.e., \( \theta_{ip}(t^p_k) \). The estimate of the control input \( u_i(t) \) can be obtained by

\[
\hat{u}_i(t) = K \sum_{j=1}^{N} a_{ij} \hat{x}_{ij}(t)
\]  

(9)

**Definition 1.** For the linear continuous-time multi-agent system (1), if \( \lim_{t \to -\infty} \| x_i(t) - x_j(t) \| = 0 \) holds, it can be said that the protocol (3) can solve the consensus problem or the multi-agent system (1) can achieve consensus under the protocol (3).

**Lemma 1** [28]. If the graph \( \mathcal{G} \) contains a directed spanning tree, zero is the simple eigenvalue of the Laplacian matrix \( \mathcal{L} \) and all the other eigenvalues have positive real parts. Otherwise, \( 1 \) is a right eigenvector associated with the zero eigenvalue.

**Lemma 2** [22]. When \( t > 0 \), \( \| e^{tA} \| \leq c_A e^{t|\lambda_{A}|} \) holds for the matrix \( A \in \mathbb{R}^{n \times n} \), where \( c_A > 0 \) and \( \lambda_A \) satisfies that if matrix \( A \) is Hurwitz, then \( \max \{ \Re(\lambda_i(A)) \} < \mu_A \leq 0 \), and otherwise \( 0 \leq \max \{ \Re(\lambda_i(A)) \} < \mu_A \).

**Lemma 3.** For the linear continuous-time multi-agent system (1) with the event-triggered consensus protocol (3) and the event function (5), if all the matrices \( A + \lambda_s(\mathcal{L})BK \) \( (s = 2, 3, \ldots, N) \) are Hurwitz, then all the matrices \( \mathcal{H}_i = I_{N-1} \otimes A + (d_i + 1_{N-1}a_i^* - A_i^*) \otimes BK \) \( (i = 1, 2, \ldots, N) \) are also Hurwitz.

**Proof.** An invertible matrix can be taken as \( S_i^{-1} = [R_i, Q_i] \), where \( R_i = [1, 0, 0, \ldots, 0] \in \mathbb{R}_{1 \times N} \) and \( Q_i \in \mathbb{R}_{(N-1) \times N} \) is a matrix which is derived by inserting \(-1_{N-1} \) before the \( i \)th column or after the \((i-1)\)th column of the identity matrix \( I_{N-1} \), i.e.,

\[
Q_i = \begin{bmatrix}
1 & 0 & \cdots & -1 & \cdots & 0 & 0 \\
0 & 1 & \cdots & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & \cdots & 0 & 1 \\
0 & 0 & \cdots & -1 & \cdots & 0 & 1
\end{bmatrix}
\]

By the definition of the Laplacian matrix \( \mathcal{L} \), it is clear that

\[
S_i^{-1} \mathcal{L} S_i = \begin{bmatrix} 0 & I_i \\ 0 & d_i + 1_{N-1}a_i^* - A_i^* \end{bmatrix}
\]  

(10)

where \( I_i = [I_{11}, I_{12}, \ldots, I_{1(i-1)}, I_{1(i+1)}, \ldots, I_{1N}] \).

It is assumed that \( \lambda_1(\mathcal{L}) = 0, \lambda_2(\mathcal{L}), \ldots, \lambda_N(\mathcal{L}) \) are the eigenvalues of the Laplacian matrix \( \mathcal{L} \). From (10), it can be seen that \( \lambda_s(\mathcal{L}) \) \((s = 2, 3, \ldots, N) \) are the eigenvalues of \( d_i + 1_{N-1}a_i^* - A_i^* \). Therefore, there exists an invertible matrix \( T_i \) such that \( d_i + 1_{N-1}a_i^* - A_i^* \) is similar to a Jordan canonical matrix.

\[
T_i^{-1}(d_i + 1_{N-1}a_i^* - A_i^*)T_i = J_i = \text{diag}(j_{i1}^*, j_{i2}^*, \ldots, j_{im}^*)
\]

(11)

where \( j_{ik}^* \) \((k = 1, 2, \ldots, m_i) \) are upper triangular Jordan blocks. The principal diagonal elements of \( j_{ik}^* \) are \( \lambda_s(\mathcal{L}) \) \((s = 2, 3, \ldots, N) \). Therefore, the following equation can be obtained

\[
(T_i \otimes I_{N-1})^{-1}((I_{N-1} \otimes A + (d_i + 1_{N-1}a_i^* - A_i^*) \otimes BK)(T_i \otimes I_{N-1}) = I_{N-1} \otimes A + J_i \otimes BK
\]

(12)

where \( I_{N-1} \otimes A + J_i \otimes BK \) is an upper triangular block matrix.

According to the properties of the Kronecker product \([1]\), it can be known that the eigenvalues of \( I_{N-1} \otimes A + J_i \otimes BK \) are given by the eigenvalues of \( A + \lambda_s(\mathcal{L})BK \) \((s = 2, 3, \ldots, N) \), i.e., the eigenvalues of the matrix \( \mathcal{H}_i \) are the same as those of \( A + \lambda_s(\mathcal{L})BK \) \((s = 2, 3, \ldots, N) \). As a result, if all the matrices \( A + \lambda_s(\mathcal{L})BK \) \((s = 2, 3, \ldots, N) \) are Hurwitz, then the matrix \( \mathcal{H}_i \) is surely Hurwitz. The proof is completed. \( \square \)

4. Analysis of the event-triggered consensus protocol

The following theorem presents the main results of this paper.

**Theorem 1.** Under the event-triggered consensus protocol (3) and the event function (5), the consensus problem of the linear continuous-time multi-agent system (1) with a directed topology \( \mathcal{G} \) can be solved without continuous communication if and only if all the matrices \( A + \lambda_s(\mathcal{L})BK \) \((i = 2, 3, \ldots, N) \) are Hurwitz, where \( \lambda_1(\mathcal{L}) \neq 0 \). In addition, the Zeno behavior does not exist.

**Proof.** (Sufficiency) From the measurement error (4), it is clear that

\[
e^{A(t-t_i)}x_i(t_k^i) + \int_{t_k^i}^{t} e^{A(t-s)}B\hat{u}_i(s)ds = x_i(t) + e_i(t)
\]  

(13)
Substituting (13) into (3) yields

$$u_i(t) = K \sum_{j=1}^{N} a_{ij} [x_i(t) + e_j(t) - x_j(t) - e_j(t)] = K[l_i x(t) + l_i e(t)]$$

(14)

where $l_i = [l_{i1}, l_{i2}, \ldots, l_{iN}]$ represents the $i$th row of the Laplacian matrix $\mathcal{L}$. $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$ and $e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T$.

Then, substituting (14) into (1) yields

$$\dot{x}_i(t) = A x_i(t) + BK[l_i x(t) + l_i e(t)]$$

(15)

Define $\delta_i(t) = x_i(t) - x_j(t)$. Then it can be known that the multi-agent system (1) will achieve consensus when $\lim_{t \to \infty} \|\delta_i(t)\| = 0$ holds. On the basis of (15), one can obtain that

$$\dot{\delta}_i(t) = \dot{x}_i(t) - \dot{x}_j(t)$$

$$= A\delta_i(t) + BK \sum_{j=1}^{N} a_{ij} [x_i(t) + e_j(t) - x_j(t) - e_j(t)] - BK \sum_{j=1}^{N} a_{ij} [x_i(t) + e_j(t) - x_j(t) - e_j(t)]$$

(16)

and (16) can be transformed to the following form

$$\dot{\delta}(t) = [l_{i1} \otimes A + (\mathcal{L}_{22} + 1_{N-1} a_{i1}^T) \otimes BK] \delta(t) + [(A_{22} + 1_{N-1} a_{1} + \mathcal{M}) \otimes BK] e(t)$$

(17)

where $\delta(t) = [\delta_1^T(t), \delta_2^T(t), \ldots, \delta_N^T(t)]^T$, $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$, $a_{i1} = [a_{12}, a_{13}, \ldots, a_{1N}]$, $a_{i} = [a_{i1}, a_{i2}, \ldots, a_{iN}]$, and

$$\mathcal{M} = \begin{bmatrix} l_{i1} & 0 & \cdots & 0 \\ l_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{i1} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(N-1) \times N}, \quad \mathcal{L}_{22} = \begin{bmatrix} l_{22} & -a_{32} & \cdots & -a_{2N} \\ -a_{22} & l_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & l_{NN} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -a_{21} & -a_{22} & \cdots & -a_{2N} \\ -a_{31} & -a_{32} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & -a_{NN} \end{bmatrix}$$

(17) can be rewritten as

$$\dot{\delta}(t) = (\Pi \delta(t) + \mathcal{V} e(t))$$

(18)

where $\Pi = l_{i1} \otimes A + (\mathcal{L}_{22} + 1_{N-1} a_{i1}^T) \otimes BK$ and $\mathcal{V} = (A_{22} + 1_{N-1} a_{1} + \mathcal{M}) \otimes BK$.

If agent $i$ is triggered, i.e., $f_i(t) \geq 0$, then its measurement error $e_i(t)$ will be reset to zero. This means that $f_i(t)$ will not cross zero and the measurement error $e_i(t)$ satisfies $\|e_i(t)\| \leq c_1 e^{-\alpha t}$ before agent $i$ is triggered. Clearly, $\|e(t)\| \leq \sqrt{N}c_1 e^{-\alpha t}$ and $\lim_{t \to \infty} \|e(t)\| = 0$ holds. Therefore, it can be seen that if the matrix $\Pi$ is Hurwitz, then system (18) can asymptotically converge to zero as $t \to \infty$, i.e., the multi-agent system (1) can achieve consensus under the consensus protocol (3) and the event function (5).

From Lemma 3, an invertible matrix can be taken as $S^{-1} = [-\mathbb{I}_{N-1} \otimes 1_{N-1} \otimes \mathbb{I}_{N-1}]$. Then $S^{-1} \mathcal{L} S = [-\mathbb{I}_{N-1} \otimes 1_{N-1} \otimes \mathbb{I}_{N-1}]$. Therefore, it can be proved, as for Lemma 3, that the eigenvalues of the matrix $\Pi$ are the same as those $A + \lambda_i(\mathcal{L}) BK(i = 2, 3, \ldots, N)$. As a result, if all the matrices $A + \lambda_i(\mathcal{L}) BK(i = 2, 3, \ldots, N)$ are Hurwitz, then the matrix $\Pi$ is surely Hurwitz. Then the system (18) can asymptotically converge to zero as $t \to \infty$, i.e., the multi-agent system (1) can achieve consensus under the consensus protocol (3) and the event function (5).

(Necessity) It is assumed that all the matrices $A + \lambda_i(\mathcal{L}) BK(i = 2, 3, \ldots, N)$ are not Hurwitz, so it is clear that the matrix $\Pi$ is not Hurwitz. If the initial value of $\delta(t)$ is not zero, then $\delta(t)$ will go to infinity as $t \to \infty$. So the multi-agent system (1) cannot achieve consensus under the consensus protocol (3) and the event function (5).

Next, the nonexistence of the Zeno behavior in the control process will be proved. From (4), it can be derived that

$$\dot{e}_i(t) = A e^{A(t-t_i)} x_i(t_i^+) + B \mathcal{U}_i(t) + \int_{t_i}^{t} A e^{A(t-s)} B \mathcal{U}_i(s) ds - \dot{x}_i(t)$$

$$= A e^{A(t-t_i)} x_i(t_i^+) + \int_{t_i}^{t} e^{A(t-s)} B \mathcal{U}_i(s) ds + B \mathcal{U}_i(t) - A \dot{x}_i(t) - B \mathcal{U}_i(t)$$

$$= A e_0(t) + BK \sum_{j=1}^{N} a_{ij} [e_0(t) - \dot{x}_j(t)] + B \mathcal{U}_i(t) \int_{t_i}^{t} e^{A(t-s)} ds - B \mathcal{U}_i(t)$$

(19)

And (14) can be rewritten as

$$u_i(t) = K(l_{i1} \otimes l_i) \delta(t) + K(l_i \otimes l_i) e(t)$$

(20)

where $l_i = [l_{i1}, l_{i2}, \ldots, l_{iN}]$. Then substituting (20) into (19) yields

$$\dot{e}_i(t) = A e_0(t) + BK \sum_{j=1}^{N} a_{ij} [e_0(t) - \dot{x}_j(t)] + B \mathcal{U}_i(t) \int_{t_i}^{t} e^{A(t-s)} ds - BK(l_{i1} \otimes l_i) \delta(t) - BK(l_i \otimes l_i) e(t)$$

(21)
From the event function (5), it can be known that \( \| e_i(t) \| \leq c_i e^{-\alpha t} \). Then it can be obtained that

\[
\| \dot{e}_i(t) \| \leq \| A \| \| e_i(t) \| + \| BK \| \sum_{j=1}^{N} a_{ij} \left( \| e^{A(t-t_j^i)} \| \| \theta_j(t_j^i) \| + \| B \hat{\Omega}_i(t) \int_{t_j^i}^{t} e^{A(s)} ds \| \right) \\
+ \| BK \| \| I \| \| L \| \| \delta(t) \| + \| BK \| \| I \| \| L \| \| e(t) \|
\]

\[
\leq \| A \| c_i e^{-\alpha t} + \| BK \| \sum_{j=1}^{N} a_{ij} \left( \| e^{A(t-t_j^i)} \| \| \theta_j(t_j^i) \| + \| B \hat{\Omega}_i(t) \int_{t_j^i}^{t} e^{A(s)} ds \| \right) \\
+ \| BK \| \| I \| \| L \| \| \delta(t) \| + \| BK \| \| I \| \| L \| \sqrt{N} c_i e^{-\alpha t}
\]

(22)

From Lemma 3, it can be known that if all the matrices \( A + \lambda_j(\mathcal{C}) BK \) \( (i = 2, 3, \ldots, N) \) are Hurwitz, then all the matrices \( \Pi \) are also Hurwitz. Then it follows from Lemma 2 that \( \| e^{\Pi(t-s)} \| \leq c_{\Pi} e^{\mu_\Pi(t-s)} \) and \( \| e^{A(t-t_j^i)} \| \leq c_A e^{\mu_A(t-t_j^i)} \), where \( \mu_\Pi < 0 \), \( c_A > 0 \), \( c_\Pi > 0 \). The solution of (18) can be obtained as

\[
\delta(t) = e^{\Pi(t-0)} + \int_{0}^{t} e^{\Pi(s)} \nu(s) ds
\]

(23)

According to Lemma 2, one has

\[
\| e^{\Pi(t-s)} \nu(s) \| \leq \beta_2 e^{\mu_A(t-s)} e^{-\alpha s}
\]

(24)

where \( \beta_2 = c_1 c_{\Pi} \sqrt{N} \| \nu \| \). Therefore, it can be derived that

\[
\| \delta(t) \| = \| e^{\Pi(t-0)} + \int_{0}^{t} e^{\Pi(s)} \nu(s) ds \| \leq \beta_1 e^{\mu_\Pi t} + \int_{0}^{t} \beta_2 e^{\mu_A(t-s)} e^{-\alpha s} ds \leq \eta_1 e^{\mu_\Pi t} + \eta_2 e^{-\alpha t}
\]

(25)

where \( \beta_1 = c_\Pi \| \delta(0) \| \), \( \eta_1 = \beta_1 + \frac{\beta_2}{\| \mu_\Pi + \alpha \|} \), and \( \eta_2 = \frac{\beta_2}{\| \mu_\Pi + \alpha \|} \). Substituting (25) into (22) yields

\[
\| \dot{e}_i(t) \| \leq \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(t-t_j^i)} \| \theta_j(t_j^i) \| + \psi_i e^{\mu_A(s-t_j^i)} + \frac{c_A}{\mu_A}
\]

(26)

where

\[
\varphi_i = \| A \| c_i + \| BK \| \| I \| \| L \| \| \delta \| \| BK \| \| I \| \| L \| \sqrt{N} c_i \\
\phi_i = \| BK \| \| I \| \| L \| \| \delta \| \\
\omega_i = \| BK \| \| I \| \| L \| \| \delta \| \\
\psi_i = \| BK \| \| B \| \| \hat{\Omega}_i(t) \| \| I \| \| L \|
\]

From (26), it can be known that

\[
\int_{t_j^i}^{t} \| \dot{e}_i(s) \| ds \leq \int_{t_j^i}^{t} \| \dot{e}_i(s) \| ds \leq \int_{t_j^i}^{t} \left( \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(t-t_j^i)} \| \theta_j(t_j^i) \| + \psi_i e^{\mu_A(s-t_j^i)} + \frac{c_A}{\mu_A} \right) ds
\]

(27)

where \( 0 < t_j^i < t \).

With the event function (5), it can be seen that when \( \int_{t_j^i}^{t} \| \dot{e}_i(s) \| ds = c_i e^{-\alpha t} \), the events will be triggered, which means that the event of agent \( i \) will not be triggered until \( \int_{t_j^i}^{t} \left( \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(s-t_j^i)} \| \theta_j(t_j^i) \| + \psi_i e^{\mu_A(s-t_j^i)} + \frac{c_A}{\mu_A} \right) ds = c_i e^{-\alpha t} \).

Let \( t_j^1 \) and \( t_j^2 \), with \( 0 < t_j^1 < t_j^2 \), be the two neighbor triggering instants of agent \( i \) and denote \( \tau = t_j^2 - t_j^1 \) as the interval time between the two triggering instants. As shown in Lemma 2, if \( A \) is Hurwitz, then we have \( \max(\text{Re}(\lambda_j(A))) < \mu_A \leq 0 \), and otherwise \( 0 \leq \max(\text{Re}(\lambda_j(A))) < \mu_A \). With (26), the following discussion unfolds depending on the value of \( \mu_A \). It is clear that

\[
\int_{t_j^1}^{t} \left( \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(s-t_j^1)} \| \theta_j(t_j^1) \| + \psi_i e^{\mu_A(s-t_j^1)} + \frac{c_A}{\mu_A} \right) ds \leq \int_{t_j^1}^{t} \left( \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(s-t_j^1)} \| \theta_j(t_j^1) \| + \psi_i e^{\mu_A(s-t_j^1)} + \frac{c_A}{\mu_A} \right) ds
\]

(28a)

\[
\int_{t_j^1}^{t} \left( \varphi_i e^{-\alpha t} + \phi_i e^{\mu_\Pi t} + \omega_i e^{\mu_A(s-t_j^1)} \| \theta_j(t_j^1) \| + \psi_i e^{\mu_A(s-t_j^1)} + \frac{c_A}{\mu_A} \right) ds
\]
Moreover, noting that $\tau$ is the solution of $\int_{t_1}^{t_1+\tau_1} e^{\alpha(t)} ds = c_1 e^{-\alpha(t_1+\tau)}$, the value of $\tau$ must be greater than or equal to $\tau_1$ and $\tau_2$ which are the solutions of (29a) and (29b), respectively.

\[
\int_{t_1}^{t_1+\tau_1} \left( \psi_i e^{-\alpha(t)} + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_1)} \quad (\mu_A \leq 0)
\]

(29a)

\[
\int_{t_1}^{t_1+\tau_2} \left( \psi_i e^{-\alpha(t)} + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_2)} \quad (\mu_A > 0)
\]

(29b)

By multiplying both sides of (29) by $e^{\alpha(t)}$, it is easy to obtain that

\[
\int_{t_1}^{t_1+\tau_1} \left( \psi_i + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_1)} \quad (\mu_A \leq 0)
\]

(30a)

\[
\int_{t_1}^{t_1+\tau_2} \left( \psi_i + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_2)} \quad (\mu_A > 0)
\]

(30b)

From the definition of $\alpha$, it can be known that there must exist a constant $\mu_1$ such that $\max \Re(\lambda_i(\Sigma)) < \mu_1 < -\alpha$. Moreover, if $A$ is not Hurwitz, then there must exist a constant $\mu_A$ such that $0 < \alpha < \max \Re(\lambda_i(A)) < \mu_A$. Therefore, $\tau_1$ and $\tau_2$ must be greater than or equal to $\tau_1^*$ and $\tau_2^*$ which are the solutions of (31a) and (31b), respectively.

\[
\int_{t_1}^{t_1+\tau_1^*} \left( \psi_i + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_1^*)} \quad (\mu_A \leq 0)
\]

(31a)

\[
\int_{t_1}^{t_1+\tau_2^*} \left( \psi_i + \phi_i e^{\alpha(t)} \right) \theta_{ij}(t_1) \left[ + \psi_i e^{\alpha(t)} + \frac{c_A}{\mu_A} \right] ds = c_1 e^{-\alpha(t_1+\tau_2^*)} \quad (\mu_A > 0)
\]

(31b)

Therefore, there must be a positive lower bound on the interval between any two neighboring event-triggered instants. The Zeno behavior is proved nonexistent. The proof is completed. □

**Remark 2.** Comparing the event-triggered consensus protocol (3) with that in [32], we can see that the protocol (3) will reduce to that in [32] if both $\int_{t_1}^{t_1+\tau_1^*} e^{\alpha(s)} B \dot{\chi}(s) ds$ and $\int_{t_1}^{t_1+\tau_2^*} e^{\alpha(s)} B \dot{\chi}(s) ds$ are eliminated from (3). Furthermore, the measurement error (4) will reduce to that in [32] if $\int_{t_1}^{t_1+\tau_1^*} e^{\alpha(s)} B \dot{\chi}(s) ds$ is removed from (4). Clearly, the method in [32] is a special case of the one proposed in this paper. In this paper, a predictor based on an artificial closed-loop system is proposed to estimate the control input of each agent. Since the control input is taken into account, we can obtain a more accurate estimate of the state. With an accurate estimate of the state, the linear multi-agent system will require far fewer event-triggered times to achieve consensus. This is why the reason why our proposed method outperforms some existing ones.

For the linear continuous-time multi-agent system (1) with the event-triggered consensus protocol (3) and the event function (5), an appropriate $K$ can be chosen to ensure that all the matrices $A + \lambda_i(\Sigma)BK$ $(i = 2, 3, \ldots, N)$ are Hurwitz by the following steps.

**Step 1.** Given $(A, B)$ in (1) is stabilizable, solve the Riccati equation $A^T P + PA - PBB^T P + I_l = 0$ and obtain a unique nonnegative definite solution $P$ such that all the eigenvalues of $A - BB^T P$ are in the open left half plane [29]. Furthermore, for any $\sigma \geq 1$ and $\omega \in \mathbb{R}$, all the eigenvalues of $A - (\sigma + j\omega)BB^T P$ $(j^2 = -1)$ are in the open left half plane [23].

**Step 2.** Select $K = -cB^T P$, where $c > \frac{1}{\min \Re(\lambda_i(\Sigma))}$, and $\lambda_i(\Sigma) \neq 0$ $(i = 2, 3, \ldots, N)$.

5. Simulation

In this section, some numerical examples are given to illustrate the effectiveness and the advantage of the method proposed in this paper.

**Example 1.** Consider a linear continuous-time multi-agent system consisting of six agents. The dynamic model of agent $i$ is described by the system (1) with

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
The communication topology among the six agents is described by a weighted graph as shown in Fig. 1. Let the initial state of the system be $x_1(0) = [0.4 \ 0.3]^T$, $x_2(0) = [0.5 \ 0.2]^T$, $x_3(0) = [0.6 \ 0.1]^T$, $x_4(0) = [0.7 \ 0]^T$, $x_5(0) = [0.8 \ -0.1]^T$, $x_6(0) = [0.4 \ -0.2]^T$. The feedback gain matrix is designed as $K = [-2.2 \ -1.1]$. The other parameters are $c_1 = 0.6$ and $\alpha = 0.4$. The Laplacian matrix of the weighted graph is

$$L = \begin{bmatrix}
3 & 0 & 0 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 
\end{bmatrix}$$

Fig. 2 shows the state trajectories of all six agents. It can be seen that the linear continuous-time multi-agent system can achieve consensus, which means the event-triggered consensus protocol proposed in this paper can solve the consensus problem of multi-agent systems effectively. In Fig. 3, the measurement error of each agent and the threshold of errors are presented. It can be seen that when the measurement error reaches the threshold, the event is triggered, and the measurement error is reset to zero.

Table 1 lists comparisons between the methods in [32,43], and the method proposed in this paper in terms of the event-triggered times of each agent. It can be seen that there are far fewer event-triggered times using the method in this paper than with the methods in [32,43].

**Example 2.** A spacecraft formation flying in low Earth orbit [21] is considered. The dynamic of each agent is given by

$$\dot{x}_i = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_i$$
Table 1: The event-triggered times of all the agents.

<table>
<thead>
<tr>
<th>Agent</th>
<th>[43]</th>
<th>[32]</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29</td>
<td>43</td>
<td>26</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>31</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

Fig. 4. Communication topology among the four satellites.

where $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\omega_0$ is the angular rate.

The formation flying scenario consists of four satellites with a communication topology described by a directed graph as shown in Fig. 4. As in [21], the state of a satellite has six components. The first three components are the spacecraft’s positions in the lateral, vertical and perpendicular axes, respectively, and the other three components are the spacecraft’s acceleration in the same right-hand coordinate system. Initial conditions are given by $x_1(0) = [200 500 300 11 12 15]^T$, $x_2(0) = [300 600 200 12 15 13]^T$, $x_3(0) = [600 400 500 13 13 14]^T$, and $x_4(0) = [500 700 400 15 11 12]^T$. Design $K$ as $\begin{bmatrix} -0.66 & -0.001 & 0 & -1.98 & 0 \\ 0 & 0 & 0 & -1.98 \\ -0.66 & 0 & 0 & -1.98 \end{bmatrix}$ and choose $\omega_0 = 0.001$, $c_1 = 300$ and $\alpha = 0.1$. The state trajectories of the four satellites and the measurement error of each satellite are shown in Figs. 5 and 6, respectively. It can be seen that the consensus is achieved asymptotically using the proposed scheme. The numbers of triggering times using the scheme in [32] and this paper are recorded in Table 2, which illustrates that with the proposed mechanism the multi-agent system requires far fewer event-triggered times to achieve consensus.
Fig. 5. State trajectories of the four satellites.

Fig. 6. Measurement errors of the satellites and the threshold of errors using the method in this paper.

Table 2
The event-triggered times of all the satellites.

<table>
<thead>
<tr>
<th>Agent</th>
<th>[32]</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>53</td>
</tr>
<tr>
<td>2</td>
<td>67</td>
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<td>3</td>
<td>70</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>69</td>
<td>53</td>
</tr>
</tbody>
</table>
6. Conclusion

This paper has investigated the event-triggered consensus for linear continuous-time multi-agent systems under a directed communication topology based on a predictor. A new event-triggered protocol has been designed based on a state predictor for the linear continuous-time multi-agent systems to achieve consensus without continuous communication. With the proposed consensus protocol, an agent only monitors its own state to determine the event-triggered instances, and the Zeno behaviors can be excluded as well. Simulation has shown that the proposed method can make the multi-agent system achieve consensus with far fewer event-triggered times, leading to a significant reduction of communication cost among agents.

In future research, the proposed method will be extended to multi-agent systems with heterogeneous dynamics or communication delays, and results in [7,16,36,37,41] will be beneficial.

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