Finite time dissipativity-based reliable control for time-varying system with delay and linear fractional uncertainties

Menghua Chen & Jian Sun

To cite this article: Menghua Chen & Jian Sun (2019) Finite time dissipativity-based reliable control for time-varying system with delay and linear fractional uncertainties, International Journal of Systems Science, 50:3, 463-478, DOI: 10.1080/00207721.2018.1562130

To link to this article: https://doi.org/10.1080/00207721.2018.1562130

Published online: 03 Jan 2019.

Submit your article to this journal

Article views: 131

View related articles

View Crossmark data
Finite time dissipativity-based reliable control for time-varying system with delay and linear fractional uncertainties

Menghua Chen and Jian Sun

School of Automation, Key Laboratory of Intelligent Control and Decision of Complex System, Beijing Institute of Technology, Beijing, People's Republic of China

ABSTRACT
This paper is concerned with the problem of finite time dissipativity-based reliable control for a time-varying system with linear fractional uncertainty (LFU) and time delay. An actuator fault model consisting of both linear and nonlinear faults is considered during the time-varying control process. By implementing an augmented time-varying Lyapunov functional and using the Wirtinger-type integral inequality, delay-dependent finite time dissipative conditions are established in forms of derivative linear matrix inequalities (DLMIs), which can guarantee the closed-loop system is finite time dissipative for all admissible uncertainties. Then, the DLMIs are transformed into a series of recursive linear matrix inequalities (RLMIs) based on the discretization method. And an algorithm is given to solve the RLMIs to obtain the state feedback gain. Simulation results demonstrate the validity of the proposed approach.

1. Introduction

Time-delays widely exist in many dynamic systems including electrical networks, biological systems and nuclear reactors. And time-delays often result in oscillations, poor performance, even instability of the system (Gu, Kharitonov, & Chen, 2003; Li & Souza, 1997). Due to the rise of telecommunications and network exchanges, time-delay systems have been generally investigated in the past decades, see, Zhang, He, Jiang, and Wu (2016), Fridman, Shaked, and Liu (2009) and Seuret and Gouaisbaut (2013). To mention a few, Sun, Liu, Chen, and Rees (2010) proposed a new type of augmented Lyapunov functional containing some triple integral terms and showed its effectiveness in reducing the conservatism of stability criteria. Using time delay approach, Freirich and Fridman (2016) considered large-scale networked control system. Li, Gao, and Gu (2016) proposed delay-dependent stability conditions for linear time-delay system based on frequency discretization.

It is necessary to point out that results in aforementioned papers are mostly concerned with Lyapunov stability that is defined over an infinite time interval. In fact, there are some cases where the transient performance or performance during some time interval is valuable to study. For example, in a terminal guidance scenario, the control goal is that the missile can intercept the target during finite time. Therefore, it makes sense to discuss the performance of the system within in a given time interval. Then the concept of finite-time stability (FTS) was proposed in Weiss and Infante (1967), which means the state of a system does not exceed a certain threshold during a fixed time interval. For a system considering both initial conditions and external constant disturbances, Amato, Ariola, and Dorato (2001) put forward the idea of finite time boundedness (FTB) by extending FTS. In recent years, considerable effort has been devoted to the issues of FTS, FTB for dynamic systems, such as, Amato, Ariola, and Cosentino (2006), Zhang, Feng, and Sun (2012) and Zhang, Shi, Nguang, and Karimi (2014). Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems were obtained in Amato, Tommasi, and Pironti (2013).

In addition, dissipativity theory denotes that the supplied energy from outside the system is greater than the increase of energy storage inside the system, which was firstly proposed in Willems (1972) and generalised subsequently in Hill and Moylan (1980). Dissipative control has been used in many dynamic systems, such as electrical networks in which part of the electrical energy is dissipated in the resistors in the form of heat, viscoelastic systems in which viscous friction is responsible for a similar loss in energy, and thermodynamic systems for which
the second law postulates a form of dissipation leading to an increase in entropy. There has been an increasing effort devoted to the study of dissipativity (Liu, Hill, & Wang, 2011; Ma, Chen, & Zhang, 2015; Tao, Lu, Shi, Su, & Wu, 2017; Zhang, Guan, & Feng, 2008). However, in these papers, authors considered dissipative performance of systems in the case of the asymptotic stability. Based on the above discussions, finite time dissipative (FTD) control is a more practical research topic and be worthy of research. There are some results on FTD analysis and control of dynamics systems, for example, Saktihivel, Saravananakumar, Kaviarasan, and Lim (2017) and Song, Niu, and Wang (2016).

It is well known that models of many control systems are partially or completely unknown, in practice, which leads to active research areas: robust stability analysis and robust control of uncertain time delay systems (Niculescu, 2001). Most literature considering uncertain system parameters always assumed the uncertainties to be norm-bounded. In this paper, we consider a more general type of uncertainty, LFU, which includes the norm-bounded uncertainty as a special case. Besides, there is an un-negligible fact that the control environment is very complex. And actuator/sensor fault is one of factors resulting in the complexity of control systems. Consequently, reliable control has attracted much attention and has been explored widely since it has the ability to maintain the acceptable and reliable performance of closed-loop systems (Tao et al., 2017; Wang, Wei, & Feng, 2009; Zhang et al., 2008). However, many existing results are mainly about actuator fault with linear multiplicative fault matrix. In fact, due to the complexity of control systems, such as dead zone or relay, sometimes actuator fault may couple with nonlinearity (Saktihivel et al., 2017; Saktihivel, Wang, Santra, & Kaviarasan, 2018). In this paper, we will consider actuator fault during the time-varying feedback control with nonlinear influence and unknown actuator fault matrix.

On the other hand, almost all the real-time practical systems possess time-varying characteristics, for example, structures and parameters of many dynamics systems may experience continuous changes caused by temperature, changes of the operating point, aging of components, etc. (Song, Niu, & Zou, 2017a). As a consequence, a large number of researches on time-varying systems emerge (Chen & Yang, 2016; Maghenem & Lorla, 2017; Phat, 2002). There are several kinds of stability on time-varying systems caused wide attention. In Mullhaupt, Bucciari, and Bonvin (2007) and Zhou (2016), asymptotic stability of time-varying systems were discussed. In Jetto and Orsini (2009), exponential stability conditions for linear time-varying systems were addressed. In Tan, Zhou, and Duan (2016), Amato, Ariola, and Cosentino (2010) and Guo et al. (2013), the problems of FTS analysis and control of time-varying systems were investigated. In Xie, Lam, and Li (2017), the problem of finite-time $H_\infty$ control was investigated for periodic piecewise linear systems by employing a time-varying control scheme. However, to the best of our knowledge, time delay phenomenon was not considered in most of the results on time-varying systems, if any, see Phat (2002), where the conditions were delay-independent. Important as it is, the problem of finite time dissipativity-based reliable control of time-varying system with LFU and delay has not been fully addressed up to now. This motivates our present study.

Inspired by the above discussion, we discuss the issue of finite time dissipativity-based reliable control for time-varying system with the consideration of time-delay, LFU and actuator failures. By constructing an augmented time-varying Lyapunov functional involving a triple integer term and using the Wirtinger-type integral inequality, delay-dependent conditions are obtained to guarantee FTD of the time-varying closed-loop system. Then, the designed state feedback controller can be obtained by solving a series of RLMIs. Finally, numerical simulations are provided to demonstrate the efficiency of the proposed results. The main contributions of this paper can be concluded as follows: (i) The finite time dissipativity-based reliable control problem is discussed for the time-varying system with LFU and time delay. The model is universally applicable in practical problems and this issue has not been discussed. (ii) A more general actuator fault model is considered in the time-varying feedback control for the first time and a new augmented time-varying Lyapunov functional involved double and triple integral is proposed to obtain delay-dependent conditions. (iii) The delay-dependent conditions in forms of DLMIs can be solved by a series of RLMIs based on a discretization method. And an algorithm is given to solve the RLMIs to obtain the finite time dissipativity-based reliable controller gain.

The remainder of this paper is organised as follows: the problem formulation and some preliminaries are introduced in Section 2. The analysis of FTB, FTD, reliable FTD controller design and discretization algorithm are given in Section 3. In Section 4, we give numerical examples to show the effectiveness of the proposed method. Section 5 concludes the paper.

Notation: Throughout this paper, $I$ and $0$ denote identity matrix and zero matrix with approximate dimensions respectively. $\mathbb{N}$ denotes the set of nature number. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. For $A \in \mathbb{R}^{n\times m}$, $A^{-1}$ and $A^T$ denote the inverse and transpose of a matrix respectively. $\lambda_{\text{min}}(A)$ means the minimal eigenvalue of...
A. For a real symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\text{sym}(A) = A + A^T$, and $A > 0 (A \geq 0)$ means that $A$ is positive defined (positive semi-defined). $C_{a,d}$ denotes the Banach space of continuous function $\phi : [-d,0] \rightarrow \mathbb{R}^n$. $\| . \|$ stands for the Euclidean norm for a vector and $\| \phi(t) \|_d = \sup_{-d \leq t \leq 0} \| \phi(t) \|$ stands for the norm of a function $\phi(t) \in C_{a,d}$. The space of square-integrable vector functions over $[0, \infty)$ is denoted by $L_2[0, \infty)$. The symbol $\ast$ means the symmetric term in a symmetric matrix.

2. Problem formulation

Consider the following time-varying system with delay and actuator failures

$$
\dot{x}(t) = \hat{A}(t)x(t) + \hat{A}_d(t)x(t - d) + B(t)u(t) + B_o(t)\omega(t)
$$

$$
z(t) = C(t)x(t) + C_d(t)x(t - d) + D_o(t)\omega(t)
$$

$$
x(t) = \phi(t), \quad t \in [-d,0]
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^p$ is the controlled output, $u(t) \in \mathbb{R}^m$ is the control input with actuator fault, $\phi(t)$ is a differentiable vector-valued initial function on $[-d,0]$, $d$ ($d > 0$) denotes the constant time delay. And the uncertain system matrices can be described as

$$
\hat{A}(t) = A(t) + \Delta A(t)
$$

$$
\hat{A}_d(t) = A_d(t) + \Delta A_d(t)
$$

(2)

where $A(t)$, $A_d(t)$, $B(t)$, $B_o(t)$, $C(t)$, $C_d(t)$, $D_o(t)$ are matrix-valued functions with compatible dimensions. $\Delta A(t)$, $\Delta A_d(t)$ are LFUs satisfying

$$
[\Delta A(t) \quad \Delta A_d(t)] = [H_1(t) \quad H_2(t)] \Delta U(t)
$$

$$
\times [E_1(t) \quad E_2(t)]
$$

(3)

where $H_1(t)$, $H_2(t)$, $E_1(t)$, $E_2(t)$ are weighting matrices with appropriate dimensions. The class of parameter uncertainty $\Delta U(t)$ satisfies

$$
\Delta U(t) = [I - U(t)]^{-1}U(t)
$$

(4)

where $J$ is a known matrix satisfying $I - JJ^T > 0$, and $U(t)$ is an unknown time-varying matrix with Lebesgue measurable elements bounded by $U^T(t)U(t) \leq I$, $\omega(t) \in L_2[0,\infty)$ is the disturbance input. For the given time interval $[0, T_c]$ and a positive scalar $b$, it satisfies

$$
\int_0^{T_c} \omega^T(t) \omega(t) dt \leq b
$$

(5)

Actuator fault is the most frequent failure in a control system. Based on this fact, we discuss actuator fault as the following form:

$$
u'(t) = Fu(t) + g(u(t))
$$

(6)

where $u(t)$ is the actuator input and $u'(t)$ is the actuator output in fault case. $F$ is the matrix of actuator effectiveness factors defined as follows:

$$
F = \text{diag}(f_1, f_2, \ldots, f_m), \quad 0 \leq f_{\bar{1}} \leq f_1 \leq \bar{f}_1 \leq 1
$$

(7)

where $f_{\bar{1}}$ and $\bar{f}_1$ ($i = 1, 2, \ldots, m$) are given constants. In the following, we introduce the following matrices

$$
F_0 = \text{diag}(f_{01}, f_{02}, \ldots, f_{0m}), \quad f_{0i} = \frac{\bar{f}_1 + f_{\bar{1}}}{2},
$$

$$
F_1 = \text{diag}(f_{11}, f_{12}, \ldots, f_{1m}), \quad f_{1i} = \frac{\bar{f}_1 - f_{\bar{1}}}{2},
$$

then, $F = F_0 + F_0 = F_0 + \text{diag}(\theta_1, \theta_2, \ldots, \theta_m)$, and $| \theta_i | \leq f_{1i}$.

The vector-valued function $g(u(t)) \in \mathbb{R}^m$ is assumed to satisfy $g^T(u(t))g(u(t)) \leq u'^T(t)LU^T(u(t))$ with a Lipschitz constant $L_g = \text{diag}(L_g1, L_g2, \ldots, L_gm)$.

Remark 2.1: The actuator fault model (6) inspired by Sakthivel et al. (2017) and Sakthivel et al. (2018) is more general than that in Zhang et al. (2008), Wang et al. (2009) and Tao et al. (2017). In Sakthivel et al. (2017), the actuator fault matrix $F$ was assumed to be known. Here, we assume it is unknown, which is more practical. In the following process of this paper, the more general actuator fault model is considered during the time-varying feedback control for the first time.

Remark 2.2: Parameters $f_{\bar{1}}$ and $\bar{f}_1$ in (7) characterise the admissible failures of the signal from the controller. When $F = 0$, it means the actuator completely fails; When $F = 1$, $g(u(t)) = 0$, it means the actuator is normal.

The following time-varying state feedback controller is considered

$$
u(t) = K(t)x(t)
$$

(8)

where $K(t)$ is the time-varying control gain to be designed.

From (1), (6) and (8), the closed-loop system can be obtained as follows

$$
\dot{x}(t) = \hat{A}_c(t)x(t) + \hat{A}_d(t)x(t - d) + B_o(t)\omega(t) + B(t)\omega(t)
$$

$$
z(t) = C(t)x(t) + C_d(t)x(t - d) + D_o(t)\omega(t)
$$

$$
x(t) = \phi(t), \quad t \in [-d,0]
$$

(9)

where $\hat{A}_c(t) = A_c(t) + \Delta A(t)$, with $A_c(t) = A(t) + B(t)FK(t)$.
Definition 2.1: For some positive constants $c_2 > c_1$, $T_c$, a positive definite matrix-valued function $\Gamma(t)$, and a disturbance input $\omega(t)$ satisfying (5), system (9) is FTB subject to $(c_1, c_2, T_c, \Gamma(t), b)$, $\forall \ t \in [0, T_c]$, if

$$\sup_{\delta \leq \theta < 0} \left\{ x^T(\theta) \Gamma(\theta) x(\theta), \dot{x}^T(\theta) \Gamma(\theta) \dot{x}(\theta) \right\} \leq c_1 \Rightarrow x^T(t) \Gamma(t)x(t) < c_2.$$ 

Remark 2.3: In Amato et al. (2001), Amato presented the quantitative definition of FTB for linear time-varying continuous system. In this paper, we extend it to time-varying continuous system with delay.

Definition 2.2: Given a scalar $a > 0$, system (9) is FTD subject to $(c_1, c_2, T_c, \Gamma(t), b, \alpha)$, if it is FTB with respect to $(c_1, c_2, T_c, \Gamma(t), b)$ and under zero initial condition, the controlled output $z(t)$ satisfies the following constrained condition with $(a, b)T_c = \int_{0}^{T_c} a^T b \, ds$

$$\langle z(t), Qz(t)T_c + 2\langle z(t), S\omega(t)T_c + \langle \omega(t), R\omega(t)T_c \rangle \rangle \alpha(\omega(t), \omega(t))T_c > \alpha(\omega(t), \omega(t))T_c \] (10)$$

Without loss of generality, we assume that $Q < 0$ and $Q = \sqrt{-Q}$.

Remark 2.4: It should be noted that the above definition of FTD is different from the conventional dissipativity theory defined over an infinite-time interval, such as (Liu et al., 2011; Ma et al., 2015; Tao et al., 2017; Zhang et al., 2008). By providing such a definition of FTD, an input-output energy-related characterisation is provided to the analysis and synthesis of systems over some a finite time interval. Moreover, the definition of FTD is fully consistent with the definition of FTB given in Amato et al. (2001). Besides, some related works can be obtained from FTD by choosing appropriate $Q, S,$ and $R$ in (10), such as:

1. Finite-time $H_\infty$ performance (Chen & Sun, 2018; Zong, Wang, Zheng, & Hou, 2015): $Q = -I, S = 0$, and $R = (\gamma + \bar{\alpha})I$.
2. Finite-time passivity (Song & He, 2015): $Q = 0, S = I$, and $R = 0$.

Lemma 2.1 (de Souza & Li, 1999; Xie, Fu, Souza, & Carlos, 1992): Let $T, M, F$ and $N$ be real matrices of appropriate dimensions with $T = T^T$ and $F^T F \leq I$, then

(a) For any scalar $\varepsilon_1 > 0$, we have

$$MFN + N^T F^T M^T \leq \varepsilon_1 M M^T + \varepsilon_1^{-1} N^T N$$

(b) $T + MFN + N^T F^T M^T < 0$, if and only if there exists a scalar $\varepsilon_2 > 0$, such that

$$T + \varepsilon_2 M M^T + \varepsilon_2^{-1} N^T N < 0.$$ 

Lemma 2.2: For a positive matrix-valued function $R(t) > 0$, the following inequality holds for all continuously differentiable function $x$ on $[a, b] \to \mathbb{R}^n$:

$$- (b - a) \int_{a}^{b} x^T(s) R(s) \dot{x}(s) \, ds \leq - (x(b) - x(a))^T R(t)(x(b) - x(a)) - 3 \Xi^T R(t) \Xi \] (11)$$

where $\Xi = x(b) + x(a) - (2/(b-a)) \int_{a}^{b} x(s) \, ds$.

Remark 2.5: In Lemma 2.2, we generalise Wirtinger-type integral inequality in Seuret and Gouaisbaut (2013) to the case where $R(t)$ is positive definite continuous matrix-valued function.

Lemma 2.3 (Zhou & Lam, 2003): Suppose that $\Delta U(t)$ is defined in (4) and some real matrices of approximate dimensions $T = T^T, M$ and $N$, then

$$T + M\Delta U(t)N + N^T \Delta U^T(t)M^T < 0,$$

if and only if there exists a scalar $\varepsilon_3 > 0$, such that,

$$\begin{bmatrix} T & M & \varepsilon_3 N^T \\ M^T & -\varepsilon_3 I & \varepsilon_3 J^T \\ \varepsilon_3 N & \varepsilon_3 J & -\varepsilon_3 I \end{bmatrix} < 0.$$ 

The aim of this paper is to design a time-varying controller in the form of (8), which can guarantee that the closed-loop system (9) under the actuator fault is not only FTB but also satisfies the given dissipative performance (10).

3. Main results

3.1. FTB analysis of the closed-loop system

This section gives sufficient conditions for FTB analysis of time-varying system (9).

Theorem 3.1: For positive constants $c_1, c_2, T_c, b, \delta, \nu$, positive definite matrix-valued function $\Gamma(t)$ and state feedback gain matrix $K(t)$, the time-varying system (9) is FTB subject to $(c_1, c_2, T_c, \Gamma(t), b), if there exist positive scalars $\varepsilon_1, \varepsilon_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$, positive definite matrix-valued functions $P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & P_3(t) \end{bmatrix} \in \mathbb{R}^{2n}$, $Q(t) \in \mathbb{R}^n$, $R(t) \in \mathbb{R}^n$, $t \in [0, T_c]$, and positive definite constant matrices $Q \in \mathbb{R}^n, W \in \mathbb{R}^n$, such that the following conditions hold

$$\Sigma(t) < 0 \] (12)$$

$$\dot{Q}(t) - Q \leq 0 \] (13)$$

$$\dot{R}(t) - R \leq 0 \] (14)$$
\[ \Gamma(t) - P_1(t) \leq 0 \]

where

\[ \lambda_{\text{max}}(\tilde{P}_1(0)) < \tilde{\lambda}_1, \quad \lambda_{\text{max}}(\tilde{P}_2(0)) < \tilde{\lambda}_2, \]
\[ \lambda_{\text{max}}(\tilde{P}_3(0)) < \tilde{\lambda}_3, \]
\[ \lambda_{\text{max}}(\tilde{Q}(0)) < \tilde{\lambda}_4, \quad \lambda_{\text{max}}(\tilde{Q}) < \tilde{\lambda}_5, \]
\[ \lambda_{\text{max}}(\tilde{R}(0)) < \tilde{\lambda}_6, \quad \lambda_{\text{max}}(\tilde{W}) < \tilde{\lambda}_7 \]
\[ \rho c_1 + b \left( 1 - e^{\delta T_c} \right) < c_2 e^{-\delta T_c} \]

\[ V_t(x(t)) = \sum_{i=1}^{4} V_i(x(t)) \]

\[ \tilde{P}_1(0) = \Gamma(0)^{-1/2} P_1(0) \Gamma(0)^{-1/2}, \]
\[ \tilde{P}_2(0) = \Gamma(0)^{-1/2} P_2(0) \Gamma(0)^{-1/2}, \]
\[ \tilde{P}_3(0) = \Gamma(0)^{-1/2} P_3(0) \Gamma(0)^{-1/2}, \]
\[ \tilde{Q}(0) = \Gamma(0)^{-1/2} Q(0) \Gamma(0)^{-1/2}, \]
\[ \tilde{R}(0) = \Gamma(0)^{-1/2} R(0) \Gamma(0)^{-1/2}, \]
\[ \tilde{W} = \Gamma(0)^{-1/2} \tilde{W} \Gamma(0)^{-1/2}. \]

**Proof:** Construct an augmented time-varying Lyapunov candidate as

\[ V(t) = \int_{0}^{t} e^{(t-s)} Q(s) x(s) ds \]
\[ + \int_{0}^{t} e^{(t-s)} Q(s) x(s) ds d\theta, \]
\[ V_3(t) = \int_{0}^{t} e^{(t-s)} Q(s) x(s) ds d\theta, \]
\[ V_4(t) = \int_{0}^{t} e^{(t-s)} Q(s) x(s) ds d\theta, \]

with \( \tilde{Q}(t) = [x^T(t) \int_{t-d}^{t} x^T(s) ds] \).

Along the trajectories of system (9), the corresponding time derivative of (18) is given by

\[ \dot{V}_1(t) = 2 \tilde{\xi}^T(t) P(t) \dot{\xi}(t) + \tilde{\xi}^T(t) \dot{P}(t) \xi(t), \]
\[ \dot{V}_2(t) = x^T(t) (Q(t) + dQ(t)) x(t) - e^{d(t-d)} Q(t) x(t-d) \]
\[ + \int_{t-d}^{t} e^{(t-s)} Q(s) x(s) ds + \delta V_2(x(t)), \]
\[ \dot{V}_3(x(t)) = dx^T(t)R(t)\dot{x}(t) \]
\[ - \int_t^t e^{\delta(t-s)}x^T(s)R(t)\dot{x}(s)ds \]
\[ + \int_0^t \int_t^t e^{\delta(t-s)}x^T(s)\dot{R}(t)\dot{x}(s)dsd\theta \]
\[ + \delta V_3(x(t)), \]
\[ \dot{V}_4(x(t)) = \frac{d^2}{2}x^T(t)W(t)\dot{x}(t) \]
\[ - \int_t^t e^{\delta(t-s)}x^T(s)W(s)\dot{x}(s)dsd\theta \]
\[ + \delta V_4(x(t)). \]

One can see that
\[ 2\xi^T(t)P(t)\dot{\xi}(t) \]
\[ = 2[x^T(t) \int_{t-d}^t x^T(s)ds \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) \\ P_2^T(t) & P_3(t) \end{bmatrix}] \]
\[ \times \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t - d) \end{bmatrix} \]
\[ = 2[x^T(t)P_1(t) + \int_{t-d}^t x^T(s)dsP_2^T(t)]\dot{x}(t) \]
\[ + 2[x^T(t)P_2(t) + \int_{t-d}^t x^T(s)dsP_3(t)] \]
\[ \times [x(t) - x(t - d)]. \]

Using (a) in Lemma 2.1, and considering \( \tilde{A}_c(t) = \tilde{A}(t) + B(t)FK(t) \), for any scalars \( \varepsilon_{1t} > 0, \varepsilon_{2t} > 0 \), it can be obtained that
\[ 2x^T(t)P_1(t)B(t)FK(t)x(t) \]
\[ \leq x^T(t)[\varepsilon_{1t}P_1(t)B(t)B^T(t)P_1^T(t) \]
\[ + \varepsilon_{2t}^{-1}(FK(t))^T(FK(t))]x(t), \]
and
\[ 2 \int_{t-d}^t x^T(s)dsP_2^T(t)B(t)FK(t)x(t) \]
\[ \leq \varepsilon_{2t} \int_{t-d}^t x^T(s)dsP_2^T(t)B(t)B^T(t)P_2(t) \int_{t-d}^t x(s)ds \]
\[ + \varepsilon_{2t}^{-1}x^T(t)(FK(t))^T(FK(t))x(t). \]

Using Lemma 2.2, one can obtain that
\[ - \int_{t-d}^t \dot{x}^T(s)R(t)\dot{x}(s)ds \]
\[ \leq -[x^T(t) - x^T(t - d)]\frac{R(t)}{d}[x(t) - x(t - d)] \]
\[ - 3\left[x^T(t) + x^T(t - d) - \frac{2}{d} \int_{t-d}^t x^T(s)ds \right] \frac{R(t)}{d} \]
\[ \times \left[x(t) + x(t - d) - \frac{2}{d} \int_{t-d}^t x(s)ds \right] \]
\[ (19) \]

Moreover, it is easy to see that for any scalar \( \nu > 0 \), the following inequality holds:
\[ \nu u^T(t)\hat{L}_g\dot{u}(t) - \nu g^T(u(t))g(u(t)) > 0 \]
\[ (20) \]

Then, it yields that
\[ \dot{V}(x(t)) = \delta V(x(t)) - \delta \omega^T(t)\omega(t) \]
\[ < \xi^T(t) \left( \begin{array}{c} \Sigma_1(t) + \Sigma_2(t)(dR(t))\Sigma_2(t) \\ + \Sigma_3(t) \left( \frac{d^2W(t)}{2} \right) \Sigma_3(t) \end{array} \right) \xi(t) \]
\[ \times \int_{t-d}^t e^{\delta(t-s)}x^T(s)(\dot{Q}(t) - Q)x(s)ds \]
\[ + \int_{t-d}^t e^{\delta(t-s)}x^T(s)(\dot{R}(t) - W)\dot{x}(s)dsd\theta, \]

where \( \xi^T(t) = [x^T(t) x^T(t - d) \int_{t-d}^t x^T(s)ds g^T(u(t)) \omega^T(t)]. \)

On account of Schur complement and (12)–(14), it gives
\[ \dot{V}(x(t)) - \delta V(x(t)) - \delta \omega^T(t)\omega(t) < 0 \]
\[ (21) \]

Multiplying (21) by \( e^{-\delta t} \), and integrating it from 0 to \( t \), it follows that
\[ V(x(t)) < e^{\delta T_c}V(x(0)) + \int_0^t e^{-\delta s}\omega^T(s)\omega(s)ds \]
\[ (22) \]

From (18) and (16), it is easy to see
\[ V(x(0)) = \xi^T(0)P(0)\xi(0) + \int_{t-d}^0 e^{-\delta s}x^T(s)Q(0)x(s)ds \]
\[ + \int_{t-d}^0 \int_{t-d}^s e^{-\delta s}x^T(s)Qx(s)ds d\theta \]
\[ + \int_{t-d}^0 \int_{t-d}^s e^{-\delta s}x^T(s)R(0)\dot{x}(s)ds d\theta \]
\[ + \int_{t-d}^0 \int_{t-d}^s e^{-\delta s}x^T(s)\dot{W}(s)\dot{x}(s)ds d\lambda d\theta \]
\[ \leq \rho \sup \left\{ x^T(\theta)\Gamma(\theta)x(\theta), \dot{x}^T(\theta)\Gamma(\theta)\dot{x}(\theta) \right\} \]
\[ \leq \rho \varepsilon_1, \]
such that
\[ V(x(t)) \leq e^{\delta T_s}(\rho c_1 + b(1 - e^{-\delta T_s})), \]
then from (17), it can be obtained that
\[ V(x(t)) < c_2. \]
On the other hand, from (15), one can see that
\[ V(x(t)) > x^T(t)P_1(t)x(t) > x^T(t)\Gamma(t)x(t). \]
Accordingly, \( x^T(t)\Gamma(t)x(t) < c_2 \) can be guaranteed for \( t \in [0, T_c] \). From Definition 2.1, system (9) is FTB. This completes the proof. \( \blacksquare \)

**Remark 3.1:** A new augmented time-varying Lyapunov functional involving a triple integral is introduced in Theorem 3.1. Meanwhile, the Wirtinger-type integral inequality is adopted, whose advantage can be achieved by coordinating with the augmented term in the Lyapunov functional (18) (Zhang et al., 2016).

### 3.2. FTD analysis for the closed-loop system

Sufficient conditions for FTD analysis of time-varying system (9) are obtained based on Theorem 3.1.

**Theorem 3.2:** For positive constants \( c_1, c_2, T_e, \delta, \alpha, b, v \), positive definite matrix-valued function \( \Gamma(t) \), state feedback gain \( K(t) \) and constant matrices \( Q, S, R \) with \( Q \leq 0, S = S^T, R = R^T \), time-varying system (9) is FTD subject to \( (c_1, c_2, T_e, \Gamma(t), b, \alpha) \), if there exist positive scalars \( \varepsilon_{11}, \varepsilon_{21}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \), positive definite matrix-valued functions \( P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & P_1(t) \end{bmatrix} \in \mathbb{R}^{2n}, Q(t) \in \mathbb{R}^n, R(t) \in \mathbb{R}^n, t \in [0, T_e] \), and positive definite constant matrices \( Q \in \mathbb{R}^n, W \in \mathbb{R}^n \), such that (13)–(17) and the following conditions hold
\[ \dot{\Sigma}(t) < 0 \quad \text{for} \quad t \in [0, T_e] \quad \text{(23)} \]
where
\[
\dot{\Sigma}(t) = \begin{bmatrix}
\Sigma_1(t) & \hat{\Sigma}_{12}(t) & \hat{\Sigma}_{13}(t) \\
* & \Sigma_{22}(t) & \hat{\Sigma}_{23}(t) \\
* & * & -I \\
\Sigma_2^T(t) & \Sigma_3^T(t) & \Sigma_5(t) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
R(t)^{-1} & 0 & 0 \\
\frac{d}{d} & \frac{2W^{-1}}{d^2} & \end{bmatrix}
\]
and
\[ \hat{\Sigma}_{12}(t) = \begin{bmatrix} P_1(t)B_0(t) - C^T(t)S \\ -C^T(t)S \\
P_2^T(t)B_0(t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} , \]
\[ \hat{\Sigma}_{13}(t) = \begin{bmatrix} C^T(t)Q \\ C^T(t)Q \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} , \]
\[ \hat{\Sigma}_{22}(t) = -2D_w^T(t)S - R + \alpha I, \quad \hat{\Sigma}_{23}(t) = D_w^T(t)Q, \]
\[ \Sigma_1(t), \Sigma_2(t), \Sigma_3(t) \text{ are defined in (12) of Theorem 3.1.} \]

**Proof:** By Theorem 3.1 and Schur complement, conditions (13)–(17) and (23) can guarantee that the closed-loop time-varying system (9) is FTB with respect to \( (c_1, c_2, T_e, \Gamma(t), b) \).

In the following, we will focus on the proof of condition (10) under zero initial condition for the closed-loop system (9). Select the same Lyapunov functional candidate as the one in Theorem 3.1 and define the following function
\[ J(t) = z^T(t)Qz(t) + 2\omega^T(t)Sz(t) + \omega^T(t)(R - \alpha I)\omega(t) \quad \text{(24)} \]
It can be seen that
\[
\dot{V}(x(t)) - \delta V(x(t)) - J(t)
\]
\[ < \xi^T(t) \left( \Sigma_1(t) + \Sigma_2^T(t)(dR(t))\Sigma_2(t) + \Sigma_3^T(t) \left( \frac{d^2W}{dt^2} \right) \Sigma_3(t) + \Omega \right) \xi(t), \]
where
\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & 0 & 0 & \Omega_{15} \\
* & \Omega_{22} & 0 & 0 & \Omega_{25} \\
* & * & 0 & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & 0 \\
\end{bmatrix},
\]
\[
\Omega_{11} = -C^T(t)QC(t), \quad \Omega_{12} = -C^T(t)QC_d(t), \quad \Omega_{15} = -C^T(t)QD_w(t) - C^T(t)S, \quad \Omega_{22} = -C^T(d(t)QC_d(t), \quad \Omega_{25} = -C^T(d(t)QD_w(t) - C^T(t)S, \quad \Omega_{55} = -D_w^T(t)QD_w(t) - 2D_2^T(t)S - (R - \alpha I). \]
By (23) and Schur complement, it is easy to see
\[
\dot{V}(x(t)) - \delta V(x(t)) - J(t) < 0 \quad (25)
\]
Integrating (25) from 0 to \(T_c\), and considering the zero initial condition, it follows that
\[
V(x(T_c)) e^{-\delta T_c} < \int_0^{T_c} e^{-\delta t} J(t) \, dt \quad (26)
\]
By the definition of \(V(x(t))\), (26) yields that
\[
\int_0^{T_c} (z^T(t)Qz(t) + 2\omega^T(t)Sz(t) + \omega^T(t)R\omega(t)) \, dt > \alpha e^{-\delta T_c} \int_0^{T_c} \omega^T(t)\omega(t) \, dt \quad (27)
\]
According to Definition 2.2 and denoting \(\alpha^* = \alpha e^{-\delta T_c}\), system (9) is FTD with respect to \((c_1, c_2, T_c, \Gamma(t), t, b, \alpha^*)\). This completes the proof. \(\Box\)

**Remark 3.2:** In Theorems 3.1 and 3.2, we consider time-varying system (9), in which the uncertainty and fault matrix are contained in \(\tilde{A}(t)\) and \(\tilde{A}_d(t)\). Therefore, Theorems 3.1 and 3.2 are the basis of robust reliable FTD controller design for the uncertain closed-loop system with actuator fault, which will be discussed in the following section.

### 3.3. Robust reliable FTD controller design for the closed-loop system

In Theorem 3.3, we discuss the problem of reliable FTD controller design problem for the closed-loop system (9) with LFUs.

**Theorem 3.3:** For positive constants \(c_1, c_2, T_c, b, \alpha, \nu, \) positive definite matrix-valued function \(\Gamma(t)\), and constant matrices \(Q, S, R\) with \(Q \leq 0, S = S^T, R = R^T\), time-varying system (9) is robust FTD subject to \((c_1, c_2, T_c, \Gamma(t), t, b, \alpha^*)\), if there exist positive scalars \(\varepsilon_t, \varepsilon_{0t}, \varepsilon_{tt}, \varepsilon_{1t}, \varepsilon_{2t}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_5, \tilde{\lambda}_6, \tilde{\lambda}_7\), positive definite matrix-valued functions \(P(t) = [P_1(t) P_2(t)] \in \mathbb{R}^{2n}\), \(Q(t) \in \mathbb{R}^n, R(t) \in \mathbb{R}^{n \times n}, t \in [0, T_c]\) and positive definite constant matrices \(Q \in \mathbb{R}^{n \times n}, W \in \mathbb{R}^{n \times n}\), such that the following conditions hold
\[
\begin{bmatrix}
\Phi(t) & \Phi_1 & \Phi_2^T \Phi_1 \\
* & -\varepsilon_{0t} I & 0 \\
* & * & -2I + \varepsilon_{0t} I
\end{bmatrix} < 0 \quad (28)
\]
\[
\dot{Q}(t) - Q(t) \leq 0 \quad (29)
\]
\[
\dot{R}(t) - W(t) \leq 0 \quad (30)
\]
\[
\Gamma(t) - P_1(t) \leq 0 \quad (31)
\]
\[
P_1(0) < \bar{\lambda}_1 \Gamma(0), P_2(0) < \bar{\lambda}_2 \Gamma(0),
\]
\[
P_3(0) < \bar{\lambda}_3 \Gamma(0),
\]
\[
Q(0) < \bar{\lambda}_4 \Gamma(0), Q < \bar{\lambda}_5 \Gamma(0),
\]
\[
R(0) < \bar{\lambda}_6 \Gamma(0), W < \bar{\lambda}_7 \Gamma(0)
\]
\[
\rho c_1 + b \left(1 - e^{\delta T_c}\right) < c_2 e^{\delta T_c} \quad (33)
\]
where
\[
\Phi(t) = \begin{bmatrix}
\Phi_1(t) & \Phi_2(t) & \Phi_3(t) \\
* & \Phi_4(t) & 0 \\
* & * & \Phi_5(t)
\end{bmatrix},
\]
\[
\Phi^1(t) = \begin{bmatrix}
\Phi_1(t) & \Phi_2(t) & \Phi_3(t) \\
* & \Phi_4(t) & \Phi_5(t) \\
* & * & \Phi_6(t)
\end{bmatrix},
\]
\[
\Phi^2(t) = \begin{bmatrix}
P_1^T(t) & 0_{1 \times 12} \\
B(t) & 0_{1 \times 12} \\
0_{1 \times 12} & 0_{1 \times 12}
\end{bmatrix},
\]
\[
\Phi^3(t) = \begin{bmatrix}
0_{12 \times 12} \\
0_{1 \times 12} \\
0_{10 \times 1}
\end{bmatrix},
\]
\[
\Phi^4(t) = \text{diag}\{ -2I + \varepsilon_{1t} I, -\varepsilon_{2t} I, -\varepsilon_{1t} I, -2I + \nu I\},
\]
\[
\Phi^5(t) = -2I + \varepsilon_{1t} I,
\]
\[
\Phi_{11}^1(t) = [\Phi_{11}^1(t)]_{1 \leq i, j \leq 9},
\]
\[
\Phi_{11}^2(t) = \text{sym}(P_1(t)A(t) + P_2(t) + \dot{P}_1(t) + Q(t) + dQ - \frac{4}{d}R(t) - \delta P_1(t),
\]
\[
\Phi_{12}^1(t) = P_1(t)A_d(t) - P_2(t) - \frac{2}{d}R(t),
\]
\[
\Phi_{22}^1(t) = -\varepsilon_{0t} Q(t) - \frac{4R(t)}{d},
\]
\[
\Phi_{13}^1(t) = A^T(t)P_2(t) + \dot{P}_2(t) + \frac{6}{d^2}R(t) - \delta P_2(t) + P_3^T(t),
\]
\[
\Phi_{23}^2(t) = A^T_d(t)P_2(t) - P_3^T(t) + \frac{6}{d^2}R(t),
\]
\[
\Phi_{33}^2(t) = \dot{P}_3(t) - \frac{12}{d^2}R(t) - \delta P_3(t),
\]
\[
\Phi_{14}^1(t) = P_1(t)B(t),
\]
\[
\Phi_{34}^2(t) = P_3(t)B(t), \Phi_{11}^3(t) = -\nu I,
\]
\[
\Phi_{15}^1(t) = P_1(t)B_{d}(t), \Phi_{35}^3(t) = P_3^T(t)B_{d}(t),
\]
Considering (2) and (3), \(13\) holds if and only if the following inequality holds

\[
\tilde{\Sigma}_{12}(t) + \Phi_{12}(t)\tilde{U}(t)\Phi_{13}^T(t) + \Phi_{13}(t)\tilde{U}^T(t)\Phi_{12}^T(t) < 0
\]

Then, from Lemma 2.3, it can be obtained that (34) holds if and only if there exists \(\varepsilon > 0\) such that

\[
\hat{\Sigma}(t) = \begin{bmatrix}
\hat{\Sigma}_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t) \\
* & 0 & 0 \\
* & * & -\tilde{\Sigma}_{33}(t)
\end{bmatrix} < 0
\]

where \(\tilde{\Sigma}_{33}(t) = \text{diag}[\varepsilon_{1}^{-1}I, \varepsilon_{2}^{-1}I]\).

By Schur complement, the following inequality can be obtained from (35)

\[
\hat{\Sigma}(t) = \begin{bmatrix}
\hat{\Sigma}_{11}(t) & \hat{\Sigma}_{12}(t) & \hat{\Sigma}_{13}(t) \\
* & 0 & 0 \\
* & * & -\hat{\Sigma}_{33}(t)
\end{bmatrix} < 0
\]

where \(\hat{\Sigma}_{11}(t) = \left[\hat{\Sigma}_{ij}^{(j)}\right]_{1\leq i,j \leq 13}\) is the same with \(\tilde{\Sigma}(t)\) excepted that the \((1,1)\) and \((3,3)\) blocks of \(\hat{\Sigma}_{11}(t)\) are replaced by \(\hat{\Sigma}_{11}(t)\). Thus we can see that (23) holds if and only if the following inequality holds

\[
\tilde{\Sigma}_{11}(t) + \Phi_{12}(t)\tilde{U}(t)\Phi_{13}^T(t) + \Phi_{13}(t)\tilde{U}^T(t)\Phi_{12}^T(t) < 0
\]

And then the state feedback controller \(K(t)\) can be derived by solving the above inequalities.

**Proof:** Considering (2) and (3), \(\hat{\Sigma}(t)\) in (23) can be written as

\[
\hat{\Sigma}(t) = \hat{\Sigma}_{11}(t) + \Phi_{12}(t)\tilde{U}(t)\Phi_{13}^T(t) + \Phi_{13}(t)\tilde{U}^T(t)\Phi_{12}^T(t)
\]

where \(\tilde{U}(t) = \text{diag}[\Delta U(t), \Delta U(t)]\), \(\hat{\Sigma}_{11}(t)\) are matrices those replace \(\tilde{A}(t), \tilde{A}_{c}(t)\) and \(\tilde{A}_{d}(t)\) in \(\hat{\Sigma}(t)\) with \(A(t), A_{c}(t)\) and \(A_{d}(t)\) respectively.

\[
\Delta U(t) = \text{diag}[\Delta U(t), \Delta U(t)]
\]

Thus we can see that (23) holds if and only if the following inequality holds

\[
\hat{\Sigma}_{11}(t) + \Phi_{12}(t)\tilde{U}(t)\Phi_{13}^T(t) + \Phi_{13}(t)\tilde{U}^T(t)\Phi_{12}^T(t) < 0
\]
from (28). Moreover, it is easy to see that (32) can be derived from (16). The proof is completed.

**Remark 3.3:** In Theorem 3.3, reliable FTD controller design method is obtained. Based on the definition of FTD, the FTD controller design method can be deduced from Theorem 3.3 by removing the dissipative performance parameters and matrices therein.

### 3.4. Discretization

Conditions in Theorem 3.3 are DLMIs, which are difficult to solve, especially the system matrices are time-varying. In the following, we will solve the above DLMIs by discretizing the time interval $[0, T_c]$. 

Inspired by the method in Shaked and Suplin (2001), Amato, Ambrosino, Cosentino, and Tommasi (2010) and Amato et al. (2010), we assume that matrix-valued functions $P(t), Q(t), R(t)$ are piecewise affine. Then, discretize the time interval $[0, T_c]$ into equally spaced subintervals with time instances $t_i (i = 0, 1, 2, \ldots, N)$, $t_0 = 0, t_N = T_c$, and

$$t_k - t_{k-1} = \Delta \tau = \frac{T_c}{N}, \quad k = 1, 2, \ldots, N.$$  

Thus, the time-varying matrix-valued functions $P(t), Q(t), R(t)$ can be expressed by linear interpolation formula as follows:

$$\Upsilon(t) = \begin{cases} \Upsilon(t_0), & t = t_0 \\ \rho_0(t) \Upsilon(t_k) + \rho_1(t) \Upsilon(t_{k-1}), & t \in (t_{k-1}, t_k]. \end{cases}$$

where $\rho_0(t) = (t - t_{k-1})/(t_k - t_{k-1}), \rho_1(t) = 1 - \rho_0(t), \Upsilon(t)$ represents $P(t), Q(t)$ and $R(t)$.

For the sake of simplicity, we denote $\Upsilon(t_k)$ as $\Upsilon_k$, then $\Upsilon(t)$ can be written as

$$\Upsilon(t) = \begin{cases} \Upsilon_0, & t = t_0 \\ \sum_{i=0}^1 \rho_i(t) \Upsilon_{k-i}, & t \in (t_{k-1}, t_k]. \end{cases}$$

On the other hand, for dealing with time-varying system matrices, we divide $(t_{k-1}, t_k)$ into $M$ segments with $t_{l-1}^k - t_{l-1} = \epsilon/M, l = 1, 2, \ldots, M$, and $t_0^k = t_{k-1}, t_M^k = t_k$. Then for $t \in (t_{l-1}^k, t_l^k)$, $A(t) = A(t_{l-1}^k + \epsilon), A_d(t) = A_d(t_{l-1}^k + \epsilon), B(t), C(t), D(t)$ can be expressed in the same way. They are denoted as $A_d^k$, $A_{dl}^k$ and so on.

Based on the above discussions, if there exist positive scalar-value sequences $\varepsilon_k, \varepsilon_{0k}, \varepsilon_{1k}, \varepsilon_{2k}$, for $k = 1, 2, \ldots, N, l = 1, 2, \ldots, M, i = 0,1$, DLMIs (28)–(31) can be expressed as follows:

\[
\begin{bmatrix}
\Phi & \Phi_1^k & (\Phi_2^k)^T F_1 \\
* & -\varepsilon_{0k} I & 0 \\
* & * & -2I + \varepsilon_{0k} I
\end{bmatrix} < 0
\]  

\[
\frac{Q_k - Q_{k-1}}{\sigma} - Q \leq 0
\]

\[
\frac{R_k - R_{k-1}}{\sigma} - W \leq 0
\]

\[
\Gamma_k^T - P_{1,k-i} \leq 0
\]
Remark 3.4: DLMIs (28)–(31) become affine continuous of the whole interval \([0, T_c]\). Therefore, if DLMIs (28)–(31) are satisfied at Step 1, Algorithm 1:

An algorithm is outlined for solving the above RLMIs.

**Algorithm 1:** Step 1. Given positive scalars \(c_2 > c_1, \alpha\) and time interval \([0, T_c]\). Set \(k = 1\), for \(l = 1, 2, \ldots, M, i = 0, 1:\n
\[
\text{find} \quad P_0, Q_0, R_0, Q, W \\
\text{subject to} \quad (32), (33).
\]

Step 2. Set \(k = k + 1\), for \(l = 1, 2, \ldots, M, i = 0, 1\), using the obtained \(P_{k-1}, Q_{k-1}, R_{k-1}, Q, W\)

\[
\text{find} \quad P_k, Q_k, R_k, K_k \\
\text{subject to} \quad (39) - (42).
\]

Step 3. If \(k \leq N\), then go to Step 2, else go to Step 4. Step 4. Stop.

Remark 3.5: It should be pointed that, the piecewise affine structures of \(P(t), Q(t), R(t)\), the left hands of DLMIs (28)–(31) become affine continuous of \(t\) over each subinterval. Therefore, if DLMIs (28)–(31) are satisfied at the extremities of the subintervals, they are satisfied over the whole interval \([0, T_c]\).

Remark 3.5: It should be pointed that, the piecewise affine restrictions of \(P(t), Q(t), R(t)\) may bring some conservatism. By increasing the number of subintervals, one can reduce the conservatism at the expense of the computational burden. In practice, the discretization rate is often taken 10–20 times faster than the bandwidth of the open and closed loop, or choose smallest possible discretisation interval which meanwhile complies with the computational resources and accuracy of the LMI solver (Shaked & Suplin, 2001).

**Remark 3.6:** In Liu, Zhou, Lei, and Tian (2015), the time interval \([0, T_f]\) was divided into \(L\) segments with the length of each segment \(h = T_f / L \geq 2d\). Thus, the segment length \(h\) cannot be sufficient small, which may bring some conservatism. In our paper, with the same bandwidth, we can adjust arbitrarily small segmentation length \(\varepsilon\) to reduce the conservatism caused by discretization. Thus our method has less conservatism from the viewpoint of computation than that in Liu et al. (2015).

### 4. Numerical examples

**Example 4.1:** Consider the time-varying system (1) with

\[
A(t) = \begin{bmatrix} 2t & 100 \\ 0 & t + 1 \end{bmatrix}, \quad A_d(t) = \begin{bmatrix} 0.1 & 0 \\ 0 & e^{-t} \end{bmatrix},
\]

\[
B(t) = \begin{bmatrix} 1 \\ 2t + 2 \end{bmatrix}, \quad B_{t_0}(t) = 1, C(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0.01t \end{bmatrix}.
\]

\[
C_d(t) = \begin{bmatrix} 0.1 & 0 \\ e^{-t} & 0 \end{bmatrix}, \quad D_{t_0}(t) = 0.1, \quad H_1(t) = \begin{bmatrix} 0 \\ t \end{bmatrix},
\]

\[
H_2(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad E_1(t) = [\sin t \quad 0],
\]

\[
E_2(t) = [0 \quad t], \quad \Gamma(t) = \mathrm{diag}[e^t, 2e^t],
\]

\[
F = \{f\}, \quad \bar{f} = 0.2, \quad \bar{f} = 0.8,
\]

\[
g(u(t)) = \sin(u(t)), \quad L_{x_0} = 0.5,
\]

\[
Q = -5, \quad S = 20, \quad R = 30,
\]

\[
c_1 = 1, \quad c_2 = 2, \quad T_c = 0.1, \quad d = 0.01,
\]

\[
\omega(t) = e^{-t}, \quad \alpha = 1, \quad \sigma = 0.01.
\]

For simulation, we give initial condition \(\phi(t) = [-0.1 - 0.5]^T\). The simulation of state measurement \(x^T(t)\Gamma(t)x(t)\) of the open-loop system is depicted in Figure 1, from which we can see that states from \((-d < \theta < 0)\) cross the border of \(x^T(t)\Gamma(t)x(t) = c_1\) within \([0, T_c]\). Using the Algorithm 1, we can get the controller gains in each subinterval and the evolutions of the controller gains are simulated in Figure 2. The control input curves of \(u(t)\) and \(u'(t)\) are given in Figure 3. The corresponding state trajectories of closed-loop system in normal case...
Figure 1. The trajectory of $x^T(t) \Gamma(t)x(t)$ for the open-loop system (Example 4.1).

Figure 2. The evolutions of the controller gain $K_1(t)$ and $K_2(t)$ (Example 4.1).

Figure 3. The trajectories of $u(t)$ and $u'(t)$ (Example 4.1).

Figure 4. The state responses of $x_1(t)$ and $x_2(t)$ for the closed-loop system in normal case (Example 4.1).

Figure 5. The state responses of $x_1(t)$ and $x_2(t)$ for the closed-loop system with actuator failure (Example 4.1).

Figure 6. The trajectory of $x^T(t) \Gamma(t)x(t)$ for the closed-loop system (Example 4.1).
Based on the discussion in Guo et al. (2013), a time-varying state-space representation can be obtained as follows

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{a}_M \\
\dot{a}_T
\end{bmatrix} = \begin{bmatrix}
-2 & -1 & 1 \\
\frac{1}{t_f - t} & \frac{-1}{\tau_M} & \frac{1}{V_c(t_f - t)} \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
\theta \\
a_M \\
a_T
\end{bmatrix}
\]

The purpose of the terminal guidance is to control the missile to intercept a target such that the miss distance is minimised, which guarantees maximal possibility of successful interception of targets within finite time interval \([0, t_e]\). The missile is destined to intercept the target when the rate of the LOS angle \(\dot{q}\) is zero.

In view of (43), the target maneuver \(a_{TC}\) is considered as external disturbance and the missile maneuver \(a_{MC}\) is regarded as control input. Considering the fact that minor failures can lead to failed interception. In this paper, we suppose the actuator failure appearing during the process. Then the relative kinematics relation between the missile and the target can be written into a linear time-varying system as follows

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + B_\omega(t)\omega(t) \\
z(t) &= C(t)x(t)
\end{align*}
\]

where

\[
A(t) = \begin{bmatrix}
-2 & -1 & 1 \\
\frac{1}{t_f - t} & \frac{-1}{\tau_M} & \frac{1}{V_c(t_f - t)} \\
0 & 0 & -1
\end{bmatrix},
B(t) = \begin{bmatrix}
0 \\
\frac{1}{\tau_M} \\
0
\end{bmatrix}, B_\omega(t) = \frac{1}{\tau_T},
C(t) = [1 \ 0 \ 0],
\end{align*}
\]

The parameters are given as \(\tau_M = 0.25/s, \tau_T = 0.25/s, \ c_1 = 1, \ c_2 = 20, \ t_e = 9.9/s, \ t_f = 10/s, \ V_c = 1500/(m \cdot s^{-1}), \ \Gamma(t) = I_3\). We assume that \(Q = -I, \ S = 0, \ R = (\alpha + \gamma^2)I\), then the dissipative control problem reduces to \(H_\infty\) control problem with an \(H_\infty\) performance index \(\gamma\).

In the process of actual operation, there exist reaction time delays, neuromuscular delays and other factors, which influence the guidance effect. Therefore, taking the time delay into account is necessary. It is assumed that the time delay is denoted as \(d (d \geq 0)\), and assume the control law \(u(t) = K(t)x(t - d)\), thus the closed-loop system

\[
z(t) = \dot{q}
\]
Figure 8. The trajectory of $x^T(t)\Gamma(t)x(t)$ of the open-loop system (Example 4.2).

Figure 9. The state responses of the closed-loop system with actuator fault (Example 4.2).

can be obtained as follows

\[ \dot{x}(t) = A(t)x(t) + B(t)FK(t)x(t - d) + B_\omega(t)\omega(t) + B(t)g(u(t)) \]

\[ z(t) = C(t)x(t) \]

Let $d = 0.01, \gamma = 1, \ F = \{f\}, f = 0.4, \bar{f} = 1, \ x_0 = [0.008 \ 4 \ 0.1]^T, \ \sigma = 0.01$, the trajectory of $x^T(t)\Gamma(t)x(t)$ of the open-loop system is simulated in Figure 8, from which we can see that it is larger than $c_2$ at the beginning. Using the method in this paper, we can obtain the reliable controller to guarantee the closed-loop system (45) be $H_\infty$ FTB. Figure 9 shows the state responses of closed-loop system with actuator failure. From Figure 9, we can see that, though the existence of actuator failure, the rate of the LOS angle $\dot{q}$ tends to be zero, and $x^T(t)\Gamma(t)x(t)$ remains within the limited range of $c_2$ during the time interval $[0, t_e]$. In conclusion, the method in this paper is effective.

It has to be mentioned that, in this paper, we consider time delay and actuator failure in the model, which have not been considered in Guo et al. (2013). Using our method, the missile can intercept the target successfully with the existence of time delay and actuator failure. Therefore, the model discussed in our paper is more universal and the our method is more practical.

5. Conclusion

The problems of FTB, FTD and robust reliable FTD for linear time-varying system with LFU and time delay have been investigated in this paper. An actuator failure model with nonlinearities has been considered. The closed-loop system is reliable FTB and satisfies the dissipative performance under the delay-dependent conditions in forms of DLMIs. By utilising discretization method, the DLMIs have been approximated by a series of RLMIs, which can be solved using the Matlab Toolbox effectively. Examples have been given to illustrate the effectiveness of the method proposed in this paper.

In this paper, the discretization method is used by dividing the time interval into equal subintervals. Can the segmentation subintervals be unequal? And can the continuous Lyapunov functional be replaced by a piecewise one? We will focus on these two questions to improve our results proposed in this paper. On the other hand, in recent years, finite time slide mode control method has been discussed for nonlinear systems in Song, Niu, and Zou (2017b, 2018). It is interesting for us to extend the finite time slide mode control method to finite-time dissipative control for time-varying systems in the near future.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This paper is supported in part by the National Natural Science Foundation of China (Nos. 61522303, U1509215, 61621063).

References


