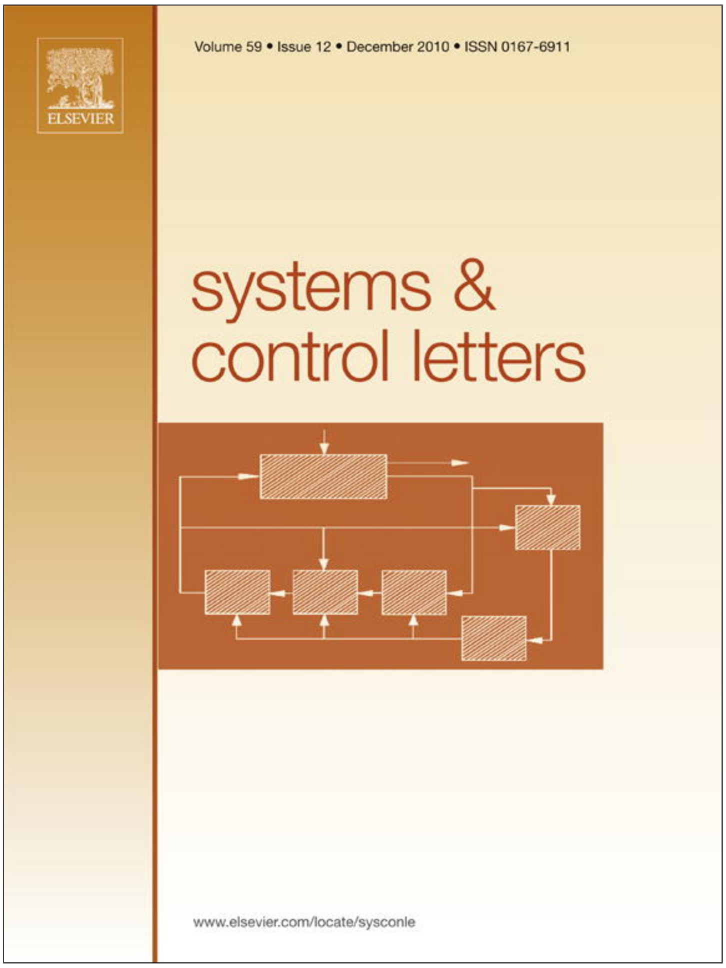
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Partial state consensus for networks of second-order dynamic agents[☆]

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ABSTRACT

This paper addresses the partial state consensus problem of multi-agent systems with second-order agent dynamics and proposes an asynchronous distributed consensus protocol for the case with switching interaction topology, time-varying delays and intermittent information transmission. "Partial state consensus" means reaching an agreement asymptotically with each other on part, but not all, of each individual's states, where the concerned states usually cannot be decoupled from the other ones. Partial state consensus has its broad applications in the coordination of multi-robot systems, distributed task management, and distributed estimation for sensor networks, etc. This paper assumes that position-like states are the only detectable information transmitted over the network and velocity-like states are the key quantities of interest, which are required to be equalized. We first give the asynchronous distributed protocol based on the delayed position-like state information and then provide its convergence result with respect to velocity-like states. It is shown that if the union of the interaction topology across the time interval with a given length always contains a spanning tree, then the proposed protocol will solve the partial state (velocity-like state) consensus problem asymptotically.

1. Introduction

Consensus theory, as a basic and fundamental research topic in distributed coordination of multi-agent systems, has received considerable attention from researchers in recent years because of its potential applications in cooperative control of unmanned air vehicles, formation control of mobile robots, design of sensor networks, swarm-based computing, etc. [1–5]. It requires that all agents reach an agreement on certain quantities of interest. The shared common value may be the anticipated attitude in multiple spacecraft alignment, position and velocity in flocking control, or processing rate in distributed task management.

Due to the complexity of network dynamics, a large amount of works assume that each individual follows a first-order differential equation [6,7]. And there is also a fraction of the literature concentrating on the networks with second-order agent dynamics. For instance, in [8], Xie and Wang considered the state consensus problem in networks of multiple double-integrators under fixed and switching topologies and designed the protocols ensuring that position-like states converge to a common static value asymptotically. In [9], Hong et al. used a set of first-order agents to track an active second-order leader, where the consensus state may be time-varying. Ren proposed and analyzed consensus algorithms for networks of double-integrators in [10,11] and second-order linear harmonic oscillators in [12] under the assumption that each agent can fully or partially access its neighbors' relative states. Along this line, finite-time agreement for multiple double-integrators was discussed in [13], where the proposed protocols are non-smooth and their consensus property was proved by the theory of finite-time "homogeneous" systems. Under these algorithms, all agents' states, including position-like and velocity-like states, will reach consensus asymptotically. However, in many practical situations, it often occurs that only a small part of state variables of each agent are the key quantities of interest that are required to be coordinated and usually cannot be decoupled from the other ones. In these cases, the concept of consensus in the traditional sense will not be suitable. "Partial state consensus" means reaching an agreement asymptotically with each other on part, but not all, of each individual's states, and in its studies, we may face many new challenges caused by the constraint of coupled agreement and nonagreement variables, such as characterizing the partial state consensus problem in the framework of coordination of full state variables, distributed estimations for interesting states of neighboring agents via the

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nonagreement coupled information, and so on. Those challenges greatly increase the difficulty of design and analysis of partial state consensus protocols. As the first step towards the general study on partial state consensus, this paper considers the simplest case and investigates the networks of second-order dynamic agents. It is assumed that position-like states are the only detectable information transmitted over the network and velocity-like states are the quantities of interest, which are required to be equalized. Next, we show several scenarios where the velocity-like state is the only quantity of interest. The first example includes the various versions of the Vicsek model [7,14,15], where the velocity consensus is a prerequisite for further study. Also in the theoretical study, in [16], Barbarossa and Scutari proposed a decentralized sensor network scheme capable of reaching a globally optimum maximum-likelihood estimate through self-synchronization of nonlinearly coupled dynamical systems. Although each node in the network they studied is a first-order dynamical system, the final agreement estimate is related to the state derivative of each sensor, namely, the authors studied the velocity-like state consensus problem in essence. In applications, one example is the congestion control of the internet. It is desirable that each router coordinates its data-processing rate to be consistent with its neighbors according to the amount of date processed in the latest time, since velocity coordination can greatly reduce the queuing delay and package loss rate, and enhance the processing efficiency. Another example is the decomposition of complex tasks. Each agent also should coordinate its processing rate to improve work efficiency.

The main contribution of this paper is to provide an effective partial state consensus control strategy, valid for the case with switching interaction topology, time-varying delays and intermittent information transmission. We first give the design result of the distributed coordination protocol based on delayed position-like state information. Then by using the tools from graph theory and nonnegative matrix theory, we show that if the union of the interaction topology across the time interval with some given length always contains a spanning tree, the partial state consensus problem will be solvable. Moreover, the studied system is an asynchronous one, which means that each agent does not necessarily approximate its neighbors' velocity-like states for its local feedback at the same time-steps by a global clock.

This paper is organized as follows. Preliminary notions are assembled in Section 2. The considered problem is formulated in Section 3. The main result is presented in Section 4 and its technical proof is postponed to Section 5. Finally, concluding remarks are summarized in Section 6.

Notations: Throughout this paper, let $\prod_{i=1}^{k} A_i = A_k A_{k-1} \cdots A_1$, denoting the left product of matrices, let I be the identity matrix and let $\mathbf{1} = [1, 1, \ldots, 1]^T$ with compatible dimensions. We write $A \geq B$ if $A - B$ is nonnegative.

2. Preliminary notions in graph theory

A directed graph $\mathcal{G}$ consists of vertex set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$ and edge set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$. The edges such as (v_i, v_i) are called *self-loops*. A *path* in directed graph $\mathcal{G}$ from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of finite vertices such that $(v_{i_l}, v_{i_{l+1}}) \in \mathcal{E}(\mathcal{G})$ for $l = 1, 2, \ldots, k-1$. Directed graph $\mathcal{G}$ is said to *have a spanning tree* if there exists a vertex, called the *root*, such that it can be connected to any other vertices through paths. The *union* of a group of directed graphs $\mathcal{G}_i, i \in \mathcal{I}$, with a common vertex set $\mathcal{V}$, is also a directed graph with the vertex set $\mathcal{V}$ and with the edge set given by $\bigcup_{i \in \mathcal{I}} \mathcal{E}(\mathcal{G}_i)$, where $\mathcal{I}$ is the index set of the group. A *weighted directed graph* $\mathcal{G}(C)$ is a directed graph $\mathcal{G}$ together with a nonnegative *weight matrix* $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff c_{ji} > 0$. And in this case, c_{ji} is called the *weight* of edge (v_i, v_j).

3. Problem formulation

The system studied in this paper consists of n autonomous agents, labeled 1 through n. All the agents share a common state space $\mathbb{R}^2$. Let x_i and v_i denote the position-like and velocity-like states of agent i respectively and suppose that agent $i, i = 1, 2, \ldots, n$, is with the following second-order dynamics

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases} \tag{1}$$

where $u_i(t)$ is a local state feedback, called *protocol*, to be designed based on the information received by agent i from its neighbors. Position-like variables x_i may represent positions, workloads, etc., and they are the only information transmitted over the network.

Suppose that each agent can communicate with some other agents which are defined as its neighbors. We use a directed graph $\mathcal{G}$ with vertex set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$ to represent the communication topology. Vertex v_i represents agent i and edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ if and only if there exists an available information channel from agent i to agent j. Because of limited detection range of agents, existence of obstacles, or external interference in signals, the communication topology is usually dynamically changing. We denote the changing topology by $\mathcal{G}(t)$.

Definition 1 (*Partial State Consensus in the Second-Order Case*). Given protocol $u_i, i = 1, 2, \ldots, n, u_i$ or this multi-agent system is said to solve *the velocity-like state consensus problem*, that is, a *partial state consensus problem*, if for any initial states, there exists a common asymptotically stable equilibrium point $v^* \in \mathbb{R}$ for all agents with respect to velocity-like states, such that $\lim_{t\to\infty} v_i(t) = v^*$ for all i.

Remark. In the case of the networks of second-order dynamic agents, governed by Eq. (1), the position-like state consensus implies the velocity-like state consensus, and thus the velocity-like state consensus is the only partial state consensus in the strict sense. So this paper focuses on the latter case only.

3.1. Velocity-like state estimation

The objective of this paper is to propose an effective protocol ensuring the solvability of the partial state consensus problem under relaxable conditions. To achieve this end, we next give a simple velocity-like state estimation strategy based on the received position-like state information.

Assume that agent i detects its neighbors' position-like states intermittently at its *update times* $t_0^i, t_1^i, \ldots, t_k^i, \ldots$, and assume that the update time sequence satisfies the following assumption:

(A1) there exist common lower and upper bounds $\check{T}_u, \hat{T}_u$ for the length of time interval between any two consecutive update times, such that $0 < \check{T}_u \leq t_{k+1}^i - t_k^i \leq \hat{T}_u$ for any $k \in \mathbb{N}$ and any $i \in \{1, 2, \ldots, n\}$.

The above assumption implies that the update time sequence is strictly increasing, unbounded and with no finite accumulation points, namely, $\lim_{k\to\infty} t_k^i = \infty$.

By the properties of the update time sequence, the studied system can be classified into two categories: the one with the property that $t_k^i = t_k^j$ for all i, j, k is called the *synchronous* system, and the other one without the preceding property is called the *asynchronous* system. This paper will focus on the asynchronous case.

At update time t_k^i, agent i may get only some of its neighbors' states because of the existence of communication time-delays. Assume that each agent is equipped with on-board memory to store information received from neighbors. The information may be with time-delays. To estimate the velocity-like states, it is further assumed that the time-delays are detectable, i.e., the information is time-stamped. Denote the available data of agent i at time t by $\mathcal{D}_i(t)$, which is composed of the information received at previous finite update times.

To be consistent with the protocol proposed in the next subsection, we give another definition of neighbors:

Definition 2 (*Neighbor Set $\mathcal{N}_i^E(t)$ and Estimation of Velocity-Like States*). Let $T_E > 0$ be given. For any $j \neq i$ and any update time t_k^i, if there exists at least one time-pair (t_1^{ijk}, t_2^{ijk}) such that

(A2) $x_j(t_1^{ijk}), x_j(t_2^{ijk}) \in \mathcal{D}_i(t_k^i)$;
(A3) $t_k^i - T_E \leq t_2^{ijk} < t_1^{ijk} \leq t_k^i$,

then agent j is called a neighbor of agent i on the time interval $[t_k^i, t_{k+1}^i)$. Denote the neighbor set of agent i in this sense at time t by $\mathcal{N}_i^E(t)$. At update time t_k^i, if $j \in \mathcal{N}_i^E(t_k^i)$, then agent i selects one such time-pair (t_1^{ijk}, t_2^{ijk}), satisfying Assumptions (A2) and (A3), and updates the estimation about the velocity-like state of agent j by

$$v_j^{E_i}(t) = \frac{x_j(t_1^{ijk}) - x_j(t_2^{ijk})}{t_1^{ijk} - t_2^{ijk}}, \quad t \in [t_k^i, t_{k+1}^i). \tag{2}$$

In the above definition, parameter T_E is a pre-given parameter, known by all agents and independent of parameters $\check{T}_u$ and $\hat{T}_u$ in Assumption (A1), and it represents the maximum allowable time-delay, guaranteeing that the data used in the designed protocol are sufficiently new. Moreover, the selection of T_E affects the number of elements in the neighbor set $\mathcal{N}_i^E(t)$ and the structure of interaction topology $\mathcal{G}^E(t)$, defined in the next subsection. And it also affects the final value of the consensus state.

In the process of data selection in approximating velocity-like states, it may happen that there exist more than one time-pairs, like (t_1^{ijk}, t_2^{ijk}), satisfying Assumptions (A2) and (A3). In this case, agent i can choose one time-pair randomly or by the *most-recent-data strategy*. In the latter case, $x_j(t_1^{ijk})$ is the most recent position-like state information about agent j in $\mathcal{D}_i(t_k^i)$ and $x_j(t_2^{ijk})$ is the most recent position-like state information satisfying Assumptions (A2) and (A3). Obviously, in this case, if $j \in \mathcal{N}_i^E(t_k^i) \cap \mathcal{N}_i^E(t_{k+1}^i)$, then $t_1^{ij,k+1} \geq t_1^{ijk}$ and $t_2^{ij,k+1} \geq t_2^{ijk}$. It can be shown by simulations that the most-recent-data strategy is more likely to result in a higher convergence rate. Notice that the choice of time-pair (t_1^{ijk}, t_2^{ijk}) in fact affects the final value of consensus state and different agents may choose different policies in a distributed manner for the choice among multiple time-pairs, satisfying Assumptions (A2) and (A3). Moreover, a lower bound requirement of $t_1^{ijk} - t_2^{ijk}$ can be added to reduce the effect of noise.

Remark. Note that estimation $v_j^{E_i}(t)$ and thus protocol (3) are only dependent on the time difference $t_1^{ijk} - t_2^{ijk}$ and state displacement across the time interval $[t_2^{ijk}, t_1^{ijk}]$. Therefore, synchronization of all agents' clocks is not a necessary condition, and t_1^{ijk} and t_2^{ijk} can be replaced by the ones decided by the local clock of agent i or j according to practical situations. However, it is required that all agents should evolve in the same time scale. From this viewpoint, the update time sequence $t_0^i, t_1^i, \ldots, \; i = 1, 2, \ldots, n$, which is decided by the global clock, can be seen as the one decided by local clocks of agents, without losing the consensus property of protocol (3).

Remark. If all agents can get their neighboring agents' position-like states without time-delays, then the assumption of "time-stamped information" can be removed. One example is the velocity consensus control of multiple robots, where $x_i(t)$ represents the position of agent i. In this case, we can suppose that at detecting times t_2^{ijk} and t_1^{ijk}, agent i can detect the relative positions of its neighboring agent j, namely, $x_j(t_2^{ijk}) - x_i(t_2^{ijk})$ and $x_j(t_1^{ijk}) - x_i(t_1^{ijk})$, respectively. If the displacement of agent i over the time interval $[t_2^{ijk}, t_1^{ijk}]$ is obtainable by agent i, then $x_j(t_1^{ijk}) - x_j(t_2^{ijk})$ can be gotten by $(x_j(t_1^{ijk}) - x_i(t_1^{ijk})) - (x_j(t_2^{ijk}) - x_i(t_2^{ijk})) + (x_i(t_1^{ijk}) - x_i(t_2^{ijk}))$.

3.2. Partial state consensus protocol

With the above preparations, we now propose the following distributed partial state consensus protocol[2]:

$$u_i(t) = \frac{1}{\sum\limits_{j \in \mathcal{N}_i^E(t_k^i)} W_{ij}(t_k^i)} \sum_{j \in \mathcal{N}_i^E(t_k^i)} W_{ij}(t_k^i)(v_j^{E_i}(t) - v_i(t)),$$
$$t \in [t_k^i, t_{k+1}^i), \tag{3}$$

where $W_{ij}(t_k^i)$ are called *weighting factors* [7], taken from a given compact set $\mathcal{W}$ consisting of positive real numbers.

Obviously, $u_i(t)$ and $v_i(t)$ are not smooth but they are piecewise differentiable with respect to time t. By this fact and by the asynchrony of update times, the differential Mid-Value Theorem does not hold, in other words, for the estimation $v_j^{E_i}(t)$ given by Eq. (2), there may not exist $t' \in [t_{ijk}^2, t_{ijk}^1]$ such that $v_j(t') = v_j^{E_i}(t)$. This shows that system (1) is not equivalent to the first-order case with time-delays studied in [4].

We end this section with a further discussion on the communication topology. We know that the communication topology $\mathcal{G}(t)$ only represents the available information flow among agents, whereas it does not indicate whether the neighbors' information is used in the feedback. The following definition of "interaction topology" reflects the relationship determined by the inter-usage of information.

Definition 3 (*Interaction Topology $\mathcal{G}^E(t)$*). The vertex set of interaction topology $\mathcal{G}^E(t)$ is $\{v_1, v_2, \ldots, v_n\}$, representing the n agents respectively, and $(v_j, v_i) \in \mathcal{E}(\mathcal{G}^E(t))$ if and only if $j \in \mathcal{N}_i^E(t)$.

4. Convergence result

This section presents the convergence result about the system under protocol (3) and its technical proof is postponed to the next section.

Theorem 4. *If there exists some $T > 0$, such that for any $t \geq 0$, the union of interaction topology $\mathcal{G}^E(t)$ over time interval $[t, t+T]$ always contains a spanning tree, and asynchronous system* (1) *satisfies Assumptions* (A1)–(A3)*, then protocol* (3) *solves the partial state (velocity-like state) consensus problem asymptotically.*

Remark. The sufficient condition provided in the above theorem is a mild and less conservative one. Here, "containing a spanning tree" is in fact to guarantee that the information of at least one agent can flow to the entire networks directly or indirectly. In the literature, the typical interaction graph conditions ensuring solvability of consensus problems under the associated proposed protocols can be generally classified into three categories. The first is that the interaction topology is always connected (in the bidirectional case) or always has a spanning tree (in the unidirectional case). This is the most conservative one. The second is that the periodical union of interaction graph is always

[2] This paper assumes that if $\mathcal{N}_i^E(t) = \emptyset$, then $u_i(t) = 0$.

connected or always contains a spanning tree. The sufficient condition stated in Theorem 4 belongs to this category. The last one is the union of all forthcoming interaction graphs contains a spanning tree and this is the mildest one for the case under time-dependent interaction topology. On the other hand, there is a tradeoff between the interaction graph condition and the basic setup of the studied system. Generally speaking, the stronger basic assumption is expected to have a relatively milder sufficient interaction graph condition. Till now, to the best of our knowledge, the only few results that can get the last mildest condition usually concern the special system in the bidirectional case, see the work of Moreau [17].

To determine whether the partial state consensus problem is solvable by the topology of information channel $\mathcal{G}(t)$, we have the following corollary:

Corollary 5. *Assume that the communication topology $\mathcal{G}(t)$ is time-invariant and contains a spanning tree, and assume that each agent can obtain its neighbors' (determined by $\mathcal{G}(t)$) position-like states with bounded time-delays for any time, that is, there exists a maximum transmission time-delay $T_{\max}$ and if there exists an information channel from agent j to agent i, then, at any time t, agent i can obtain agent j's position-like state, denoted by $x_j(t')$, with the property that $0 \le t - t' \le T_{\max}$. Then there exists an available distributed control rule in the form of protocol (3), satisfying Assumptions (A1)–(A3), and it solves the partial state (velocity-like state) consensus problem asymptotically.*

Proof. To prove the result, it suffices to find a possible distributed control rule, satisfying the assumptions assumed by Theorem 4.

Suppose that the maximum transmission time-delay is $T_{\max}$ and let the update time sequence of agent $i, i = 1, 2, \dots, n$, be $t_0^i, t_1^i = t_0^i + T_{\max}, \dots, t_{k+1}^i = t_k^i + T_{\max}, \dots$. Then it satisfies Assumption (A1) with $\check{T}_u = \hat{T}_u = T_{\max}$. We further assume that agent i measures all its neighbors' position-like state information in the time intervals (t_{k-1}^i, t_k^i), $k = 1, 2, \dots$. For j with $(v_j, v_i) \in \mathcal{E}(\mathcal{G})$, denote the obtained information related to agent j in time interval (t_{k-1}^i, t_k^i) by $x_j(t_k^{ij})$. Then $t_{k-2}^i < t_k^{ij} < t_k^i$ and thus $t_k^{ij} > t_{k-2}^{ij}$. For $t \in [t_k^{ij}, t_{k+1}^{ij})$, $k \ge 4$, let

$$v_j^{E_i}(t) = \frac{x_j(t_k^{ij}) - x_j(t_{k-2}^{ij})}{t_k^{ij} - t_{k-2}^{ij}}.$$

Then it is well defined. Since $t_{k-4}^i < t_{k-2}^{ij} < t_k^{ij} \le t_k^i$, we have that t_k^{ij} and t_{k-2}^{ij}, in the place of t_1^{ijk} and t_2^{ijk}, satisfy Assumptions (A2) and (A3) with $T_E = 4T_{\max}$. Furthermore, the above assumption also implies that for $t \ge \max_{ij} t_4^{ij}$, $\mathcal{E}(\mathcal{G}^E(t)) = \mathcal{E}(\mathcal{G}(t))$. Therefore, under the above proposed control rule, the system solves the partial state consensus problem asymptotically. □

5. Technical proof

This section performs the convergence analysis on asynchronous system (1) based on the nonnegative matrix approach, which is an effective way to show the consensus property of multi-agent systems with switching topology and time-varying delays. The first subsection summarizes some key lemmas, established in [4,5,18]. They describe the convergence property of the product of a compact set of *SIA* (Stochastic, Indecomposable and Aperiodic) matrices and give relaxable conditions ensuring a stochastic matrix to be an *SIA* matrix, respectively. The second subsection collects all event times and merges them into a single ordered time sequence, denoted by $t_0, t_1, \dots, t_k, \dots$. The evolution of velocity-like states and their estimation values is studied with respect to the above newly defined time sequence. This approach is called the "Analytic Synchronization" method in [19]. Then we define a set of new variables $v_i^A(t_{k+1})$ by $(x_i(t_{k+1}) - x_i(t_k))/(t_{k+1} - t_k)$, $i = 1, 2, \dots, n$. It is shown that the estimated velocity-like states $v_j^{E_i}(\cdot)$ can be represented by a convex combination of variables $v_j^A(\cdot)$. By investigating the relationship among these variables, we introduce an augmented $(m+1)n$-dimensional state variable $y(k)$, and then transform the continuous-time system (1) into its equivalent discrete-time system (7), where m is a nonnegative integer, determined by Assumptions (A1) and (A3). Thus the partial state consensus problem can be treated as a full state consensus problem equivalently. However, the state matrix $\Xi(k)$ of the discrete-time system has its special structure, which is different from that of the existing ones investigated in the literature [4,5,7,15,18]. Fortunately, by constructing and characterizing two compact sets $\mathcal{M}$ and $\mathcal{H}$ in the third subsection, which include all possible state matrices of system (7) and all possible products of a finite number of state matrices at consecutive time-steps of system (7), respectively, we can apply the key lemmas, presented in the first subsection, and get the convergence result. Here, we emphasize that although the proof steps are similar to that taken in [4], in other words, they are all based on the presented key lemmas, the differences between the contributions of the two papers are also obvious. First, they study two distinct kinds of problems. The feasibility of the proof with the help of Lemmas 6–8 owes much to the skillful choice of state variable $y(k)$. Second, the proof details are also different. This paper gives the only arguments that are needed to be clarified and omits the obvious ones that can be learnt from other literature.

5.1. Key lemmas

In this subsection, we first give the definition of *SIA* matrix and then list three important lemmas, which are useful in proving the main result.

A stochastic matrix A is called indecomposable and aperiodic (*SIA*) if there exists a column vector ν such that $\lim_{k\to\infty} A^k = \mathbf{1}\nu^T$.

Lemma 6 ([5, Lemma 5]). *Let $\mathcal{A}$ be a compact set, consisting of $n \times n$ SIA matrices. If for any k and any $A_1, A_2, \dots, A_k \in \mathcal{A}$ (repetitions permitted), $\prod_{i=1}^k A_i$ is SIA, then for any given infinite matrix sequence $A_1, A_2, \dots, A_k, \dots$ (repetitions permitted), there exists a column vector ν such that*

$$\lim_{k\to\infty} \prod_{i=1}^{k} A_i = \mathbf{1}\nu^T.$$

Lemma 7 ([18, Lemma 1]). *Let A be a stochastic matrix. If $\mathcal{G}(A)$ contains a spanning tree with the property that the associated root vertex has a self-loop in $\mathcal{G}(A)$, then A is SIA.*

Lemma 8 ([4, Lemma 8]). *Let $A_0, A_1, \dots, A_m$ be $n \times n$ nonnegative matrices, let*

$$D = \begin{bmatrix} A_0 & A_1 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+1)n\times(m+1)n}$$

let

$$Q_0 = \begin{bmatrix} I & & & & \\ I & & & \mathbf{0} & \\ & I & & & \\ \mathbf{0} & & \ddots & & \\ & & & I & 0 \end{bmatrix}_{(m+1)n\times(m+1)n}$$

and let $Q_k = D + Q_0{}^k$ for any $k \in \{1, 2, \dots, m\}$. If $\mathcal{G}(\sum_{i=1}^m A_i)$ contains a spanning tree, then $\mathcal{G}(Q_k)$ also contains a spanning tree with the property that the associated root vertex has a self-loop in $\mathcal{G}(Q_k)$.

5.2. Equivalent representation

In this subsection, we employ the "Analytic Synchronization" method and get the discrete-time Eq. (7), that is, an equivalent representation of the original continuous-time system. The basic idea of "Analytic Synchronization" is to study the asynchronous system by using a suitably defined discrete-time synchronous system, evolving on the collection of event times of all original subsystems [19].

First, for symbolic simplicity, we generalize the definition of $v_j^{E_i}(t)$ and introduce a weight matrix $A(t) = [a_{ij}(t)] \in \mathbb{R}^{n\times n}$, where $v_j^{E_i}(t)$ is generalized by

1. if $j \in \mathcal{N}_i^E(t)$, then $v_j^{E_i}(t)$ is defined by Eq. (2);
2. if $j = i$, then $v_j^{E_i}(t) = v_i(t)$;
3. in other cases, $v_j^{E_i}(t) = 0$,

and $A(t)$ is defined by

1. if $\mathcal{N}_i^E(t_k^i) \neq \emptyset$,

$$a_{ij}(t) = \begin{cases} \dfrac{W_{ij}(t_k^i)}{\sum\limits_{s\in\mathcal{N}_i^E(t_k^i)} W_{is}(t_k^i)}, & j \in \mathcal{N}_i^E(t_k^i) \\ 0, & \text{otherwise,} \end{cases} \quad t \in [t_k^i, t_{k+1}^i)$$

2. if $\mathcal{N}_i^E(t_k^i) = \emptyset$,

$$a_{ij}(t) = \begin{cases} 1, & j = i \\ 0, & \text{otherwise,} \end{cases} \quad t \in [t_k^i, t_{k+1}^i).$$

By the above definition, $v_j^{E_i}(t)$, $i \neq j$, is a piece-wise constant function of time t, and if ignore the weight of each edge and self-loops in $\mathcal{G}(A(t))$, then $\mathcal{G}(A(t))$ and $\mathcal{G}^E(t)$ represent the same interaction topology. Noticing that $W_{ij}(t_k^i) \in \mathcal{W}$, we get that all possible $A(t)$ constitute a compact set $\mathcal{A}$ and their nonnegative entries are not less than $\min\{w : w \in \mathcal{W}\}/((n-1)\max\{w : w \in \mathcal{W}\})$.

Now collect all time $\{t_k^i, t_1^{ijk}, t_2^{ijk} : i = 1, 2, \dots, n, j \in \mathcal{N}_i^E(t_k^i), k \in \mathbb{N}\}$ and relabel the nonnegative elements of them by $t_0, t_1, \dots, t_k, \dots$, in increasing order such that $t_0 = 0$, $t_k < t_{k+1}$. Here, we assume that $t_0^i = 0$ for all $i \in \{1, 2, \dots, n\}$. Indeed, without this assumption, we can consider the dynamics of agents after time $\max_i t_0^i$ and get the same convergence result. For simplicity, denote $t_{k+1} - t_k$ by τ_k and denote $(x_i(t_{k+1}) - x_i(t_k))/(t_{k+1} - t_k)$ by $v_i^A(t_{k+1})$ (superscript "A" means "average velocity"), for $k \in \mathbb{N}$, $i = 1, 2, \dots, n$.

Next we study the evolution of variables $v_i(t_k)$ and $v_i^A(t_k)$ with respect to k. We will prove that their reaching an agreement implies the solvability of the partial state consensus problem. Solving Eq. (3) gives that

$$\begin{cases} v_i(t) = \mathrm{e}^{-(t-t_k^i)} v_i(t_k^i) + (1 - \mathrm{e}^{-(t-t_k^i)}) \sum\limits_{j=1}^n a_{ij}(t_k^i) v_j^{E_i}(t_k^i) \\ \dfrac{x_i(t) - x_i(t_k^i)}{t - t_k^i} = \dfrac{1 - \mathrm{e}^{-(t-t_k^i)}}{t - t_k^i} v_i(t_k^i) \\ \qquad + \left(1 - \dfrac{1 - \mathrm{e}^{-(t-t_k^i)}}{t - t_k^i}\right) \sum\limits_{j=1}^n a_{ij}(t_k^i) v_j^{E_i}(t_k^i) \end{cases} \tag{4}$$

where $t_k^i < t \le t_{k+1}^i$. It can be observed that $0 < (1 - \mathrm{e}^{-(t-t_k^i)})/(t - t_k^i) < 1$ for any $t_k^i < t \le t_{k+1}^i$. The above equation implies that

$$\begin{cases} v_i(t_{k+1}) = \mathrm{e}^{-\tau_k} v_i(t_k) + (1 - \mathrm{e}^{-\tau_k}) \sum\limits_{j=1}^n a_{ij}(t_{s_i}^i) v_j^{E_i}(t_{s_i}^i) \\ v_i^A(t_{k+1}) = \dfrac{1 - \mathrm{e}^{-\tau_k}}{\tau_k} v_i(t_k) \\ \qquad + \left(1 - \dfrac{1 - \mathrm{e}^{-\tau_k}}{\tau_k}\right) \sum\limits_{j=1}^n a_{ij}(t_{s_i}^i) v_j^{E_i}(t_{s_i}^i) \end{cases} \tag{5}$$

where $s_i \in \mathbb{N}$ such that $t_{s_i}^i \le t_k < t_{k+1} \le t_{s_i+1}^i$.

This part gives an expression $v_j^{E_i}(t_{s_i}^i)$ in Eq. (5) in terms of $v_j^A(\cdot)$. Suppose that $j \in \mathcal{N}_i^E(t_{s_i}^i)$, $t_2^{ijs_i} = t_{l_i}$ and $t_1^{ijs_i} = t_{p_i}$. By Assumptions (A1) and (A3), there exists an $m \in \mathbb{N}$, independent of i, j, k (cf. Lemma 2 in [4]), such that $t_{k-m} \le t_{l_i} < t_{p_i} \le t_{s_i}^i$ for $k \ge m$. Then

$$\begin{aligned} v_j^{E_i}(t_{s_i}^i) &= \frac{x_j(t_{p_i}) - x_j(t_{l_i})}{t_{p_i} - t_{l_i}} \\ &= \frac{\tau_{p_i-1} v_j^A(t_{p_i}) + \tau_{p_i-2} v_j^A(t_{p_i-1}) + \cdots + \tau_{l_i} v_j^A(t_{l_i+1})}{\tau_{p_i-1} + \tau_{p_i-2} + \cdots + \tau_{l_i}} \end{aligned} \tag{6}$$

which means that $v_j^{E_i}(t_{s_i}^i)$ is a convex combination of $v_j^A(t_{l_i+1})$, $v_j^A(t_{l_i+2}), \dots, v_j^A(t_{p_i})$. From the fact that $p_i - l_i \le m$, it follows that some of its coefficients are not less than $1/m$.

To represent the evolution Eq. (5) in matrix form, we introduce the augmented state variable $y(k) = [v(t_k)^T, v^A(t_k)^T, v^A(t_{k-1})^T, \dots, v^A(t_{k-m+1})^T]^T$ for $k \ge m$, where $v(t_k) = [v_1(t_k), v_2(t_k), \dots, v_n(t_k)]^T$ and $v^A(t_k) = [v_1^A(t_k), v_2^A(t_k), \dots, v_n^A(t_k)]^T$. Combining Eqs. (5) and (6) yields that

$$y(k+1) = \Xi(k) y(k), \tag{7}$$

where $\Xi(k) \in \mathbb{R}^{(m+1)n\times(m+1)n}$ is defined in Box I and $A_\gamma(k) = [a_{ij}^\gamma]$, $\gamma \in \{0, 1, \dots, m\}$, are defined by

1. if $\mathcal{N}_i^E(t_{s_i}^i) \neq \emptyset$,

$$a_{ij}^\gamma = \begin{cases} \dfrac{\tau_{k-\gamma} a_{ij}(t_{s_i}^i)}{\tau_{p_i-1} + \tau_{p_i-2} + \cdots + \tau_{l_i}}, \\ \quad j \in \mathcal{N}_i^E(t_{s_i}^i),\ k - p_i + 1 \le \gamma \le k - l_i \\ 0, \quad \text{otherwise} \end{cases}$$

2. if $\mathcal{N}_i^E(t_{s_i}^i) = \emptyset$,

$$a_{ij}^\gamma = \begin{cases} 1, & j = i, \gamma = 0 \\ 0, & \text{otherwise.} \end{cases}$$

In the following, the matrix given in Box I, with the above structure, is denoted by $M(\tau_k, A_0(k), A_1(k), \dots, A_m(k))$ to indicate its dependence on parameters $\tau_k, A_0(k), A_1(k), \dots, A_m(k)$.

By the above definition, it can be easily obtained that

Lemma 9. *1. $\sum_{\gamma=0}^m A_\gamma(k) = A(t_k)$ and thus $\Xi(k)$ is stochastic;*
2. if $j \in \mathcal{N}_i^E(t_k^i)$, there exists $1 \le \gamma \le m$, such that $a_{ij}^\gamma \ge \min\{w : w \in \mathcal{W}\}/(m(n-1)\max\{w : w \in \mathcal{W}\})$.

Remark. Eq. (7) can be seen as the dynamical equation of a discrete-time multi-agent system consisting of $(m+1)n$ agents under the time-varying interaction topology $\mathcal{G}(\Xi(k))$, and variable $y(k)$ is the column vector, stacked with the states of the $(m+1)n$ agents. These states are the velocity-like states $v_i(t_k)$, $i = 1, 2, \dots, n$, and the average velocity $v_i^A(t_k), v_i^A(t_{k-1}), \dots, v_i^A(t_{k-m+1})$, $i = 1, 2, \dots, n$. In Lemma 11, we will show that their convergence to a consensus state as $k \to \infty$ will lead to that $v_i(t)$, $k = 1, 2, \dots, n$, reach consensus asymptotically as $t \to \infty$. Consensus problems of discrete-time multi-agent systems were widely studied by researchers.

$$\Xi(k)=\begin{bmatrix} e^{-\tau_k}I+(1-e^{-\tau_k})A_0(k) & (1-e^{-\tau_k})A_1(k) & \cdots & (1-e^{-\tau_k})A_{m-1}(k) & (1-e^{-\tau_k})A_m(k)\\ \frac{1-e^{-\tau_k}}{\tau_k}I+(1-\frac{1-e^{-\tau_k}}{\tau_k})A_0(k) & (1-\frac{1-e^{-\tau_k}}{\tau_k})A_1(k) & \cdots & (1-\frac{1-e^{-\tau_k}}{\tau_k})A_{m-1}(k) & (1-\frac{1-e^{-\tau_k}}{\tau_k})A_m(k)\\ & I & & & \\ \mathbf{0} & & \ddots & & \mathbf{0}\\ & & & I & \end{bmatrix}$$

Box I.

However, this discrete-time model cannot be covered by the existing ones, such as those studied in [17,7,20], because the diagonal entries of state matrix $\Xi(k)$ are not all larger than zero and parameters τ_k may take any value in $(0, \hat{T}_u]$, see [4] for detailed discussions.

Definition 10 (*Full State Consensus*)**.** The discrete-time system (7) is said to solve a *(full state) consensus problem* if for any initial state, there exists an asymptotically stable equilibrium point $\mathbf{1}y^*$, $y^* \in \mathbb{R}$, such that $\lim_{k\to\infty} y(k) = \mathbf{1}y^*$, in other words, $\lim_{k\to\infty} v_i(t_k) = \lim_{k\to\infty} v_i^A(t_k) = y^*$, $i = 1, 2, \ldots, n$.

The following lemma states the relationship between continuous-time system (1) and discrete-time system (7).

Lemma 11. *System (1) solves a partial state (velocity-like state) consensus problem if and only if system (7) solves a full state consensus problem.*

Proof. The necessity is obvious and we only prove the sufficiency.

Suppose that $\lim_{k\to\infty} v_i(t_k) = \lim_{k\to\infty} v_i^A(t_k) = v^*$ for all $i = 1, 2, \ldots, n$. Thus by Eqs. (4) and (6) and Assumption (A1), $\lim_{t\to\infty} v_i(t) = v^*$. □

The next subsection will prove Theorem 4 by showing system (7) solves a full state consensus problem.

5.3. Convergence analysis of system (7)

This subsection consists of three parts. The first part gives an equivalent representation of the condition assumed in Theorem 4. The second part characterizes the properties of state matrix $\Xi(k)$ of system (7) by two compact matrix sets $\mathcal{M}$ and $\mathcal{H}$. The last part gives the proof of Theorem 4.

First, the following lemma restates the condition provided in Theorem 4 in an equivalent way.

Lemma 12. *If there exists $T > 0$, such that for any $t \geq 0$, the union of interaction topology $\mathcal{G}^E(t)$ across the time interval $[t, t+T]$ always contains a spanning tree, then there exist a positive integer h and a positive real number T_h with the following property:*

for any $k \in \mathbb{N}$, there exists a subset of $\{t_k, t_{k+1}, \ldots, t_{k+h-1}\}$, denoted by $\mathcal{T}_k$, such that the union of $\mathcal{G}^E(t)$ on $\mathcal{T}_k$ contains a spanning tree, and for any $t_l \in \mathcal{T}_k$, $T_h \leq \tau_l \leq \hat{T}_u$.

Proof. It is a straightforward consequence of Assumptions (A1) and (A3) and the details are omitted, see Lemma 9 in [4] for similar discussions. □

In order to make use of Lemma 6 to perform the convergence analysis, we next construct two compact sets $\mathcal{M}$ and $\mathcal{H}$.

The first compact set $\mathcal{M}$ includes all possible state matrices of system (7), which is defined by

$$\mathcal{M} = \Big\{M(\varsigma, N_0, N_1, \ldots, N_m) : \\ 0 \leq \varsigma \leq \hat{T}_u, N_0, N_1, \ldots, N_m \text{ are nonnegative}, \\ \text{and there exists some } A' \in \mathcal{A},\ \text{such that} \\ N_0 + N_1 + \cdots + N_m = A'\Big\}$$

where we use the convention $(1-e^{-\varsigma})/\varsigma|_{\varsigma=0} = \lim_{\varsigma\to 0}(1-e^{-\varsigma})/\varsigma = 1$. The second compact set $\mathcal{H}$ includes all possible products of h state matrices at consecutive time-steps of system (7), which is defined by

$$\mathcal{H} = \Big\{\prod_{i=1}^{h} M(\varsigma^i, N_0^i, N_1^i, \ldots, N_m^i) : \\ M(\varsigma^i, \cdot) \in \mathcal{M},\ \text{and there exists a subset of} \\ \{1, 2, \ldots, h\},\ \text{denoted by } \mathcal{T},\ \text{such that for any} \\ k \in \mathcal{T}, T_h \leq \varsigma^k \leq \hat{T}_u \text{ and } \mathcal{G}\Big(\sum_{i\in\mathcal{T}}\sum_{j=0}^{m} N_j^i\Big) \\ \text{contains a spanning tree}\Big\}$$

where h and T_h are given in Lemma 12.

Lemma 13. 1. *$\mathcal{M}$ and $\mathcal{H}$ are compact sets, and for any $k \geq m$, $\Xi(k) \in \mathcal{M}$ and $\prod_{l=k}^{k+h-1}\Xi(l) \in \mathcal{H}$;*
2. *for any $H \in \mathcal{H}$, H is SIA, and for any $k \in \mathbb{N}$, if $H_1, H_2, \ldots, H_k \in \mathcal{H}$ (repetitions permitted), then $\prod_{i=1}^{k} H_i$ is SIA.*

Proof. (a) The compactness of $\mathcal{M}$ follows from the fact that set $\mathcal{A}$ and interval $[0, \check{T}_u]$ are compact; the compactness of $\mathcal{H}$ follows from that

1. $\mathcal{M}$ is compact;
2. all possible choices of $\mathcal{T}$ and the spanning tree are finite;
3. the product of finite matrices is a continuous function;
4. all the nonnegative entries of matrices in $\mathcal{A}$ are lower bounded by $\min\{w : w \in \mathcal{W}\}/((n-1)\max\{w : w \in \mathcal{W}\})$.

(b) The conclusion that $\Xi(k) \in \mathcal{M}$ follows directly from the definitions of $\Xi(k)$ and $\mathcal{M}$, and Lemma 9. By Lemmas 9 and 12, $\mathcal{G}(\sum_{l\in\mathcal{T}_k}\sum_{\gamma=0}^{m}(A_\gamma(l)))$ contains a spanning tree, and thus $\prod_{l=k}^{k+h-1}\Xi(l) \in \mathcal{H}$.

(c) Let $H = \sum_{i=1}^{h} M(\varsigma^i, N_0^i, N_1^i, \ldots, N_m^i)$ and $\mathcal{T}$ be the associated subset of $\{1, 2, \ldots, h\}$ such that for any $k \in \mathcal{T}, T_h \leq \varsigma^k \leq \hat{T}_u$ and $\mathcal{G}(\sum_{i\in\mathcal{T}}\sum_{j=0}^{m} N_j^i)$ contains a spanning tree. Let Q_0 be the same as the Q_0 in Lemma 8, let

$$D_i = \begin{bmatrix} N_0^i & N_1^i & \cdots & N_m^i \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and let $\varepsilon = \inf\{e^{-\varsigma}, (1-e^{-\varsigma})/\varsigma : \varsigma \in (0, \hat{T}_u]\}$. Then $\varepsilon > 0$ and

$$\begin{aligned}\prod_{i=1}^{h} M(\varsigma^i, \cdot) &\geq \prod_{i=1}^{h}\big(\varepsilon Q_0 + (1-e^{-\varsigma^i})D_i\big)\\ &\geq \varepsilon^h {Q_0}^h + \varepsilon^{h-1}\sum_{i=1}^{h}(1-e^{-\varsigma^i}){Q_0}^{i-1}D_i{Q_0}^{h-i}\\ &\geq \min\{\varepsilon^h, \varepsilon^{h-1}(1-e^{-T_h})\}\Big({Q_0}^h + \sum_{i\in\mathcal{T}} D_i{Q_0}^{h-i}\Big)\end{aligned}$$

where the last inequality follows from ${Q_0}^{i-1}D_i \geq D_i$ and from the fact that $T_h \leq \varsigma^i \leq \hat{T}_u$ for $i \in \mathcal{T}$.

Let the first n rows of $D_i{Q_0}^{h-i}$ be $\left[B_0^i, B_1^i, \dots, B_m^i\right]$, where $B_j^i \in \mathbb{R}^{n\times n}$, $j = 0, 1, \dots, m$. Then $\sum_{j=1}^m B_j^i = \sum_{j=0}^m N_j^i$ and thus $\sum_{i\in\mathcal{T}} \sum_{j=0}^m B_j^i = \sum_{i\in\mathcal{T}} \sum_{j=0}^m N_j^i$. Since $\mathcal{G}\left(\sum_{i\in\mathcal{T}} \sum_{j=0}^m N_j^i\right)$ contains a spanning tree, $\mathcal{G}\left(\sum_{i\in\mathcal{T}} \sum_{j=0}^m B_j^i\right)$ also contains a spanning tree. Let $D \in \mathbb{R}^{(m+1)n\times(m+1)n}$, with the same first n rows as $\sum_{i\in\mathcal{T}} D_i {Q_0}^{h-i}$ and with the other rows being zeros. Then

$$\prod_{i=1}^h M(\varsigma^i, \cdot) \geq \min\{\varepsilon^h, \varepsilon^{h-1}(1-\mathrm{e}^{-T_h})\}({Q_0}^h + D).$$

By Lemma 8 and the fact that ${Q_0}^k = {Q_0}^m$ if $k \geq m$, $\mathcal{G}({Q_0}^h + D)$ contains a spanning tree with the property that the associated root vertex has a self-loop, and so is $\mathcal{G}(H)$. Since H is stochastic, by Lemma 7, H is *SIA*.

By the same arguments, we have that for any $H_1, H_2, \dots, H_k \in \mathcal{H}$, $\prod_{i=1}^k H_i$ is *SIA*. □

Proof of Theorem 4.

This part only proves that system (7) solves a full state consensus problem.

It follows from Lemmas 6 and 13 that there exists $\nu \in \mathbb{R}^{(m+1)n}$ such that

$$\lim_{p\to\infty} \prod_{l=0}^{ph-1} \Xi(m+l) = \mathbf{1}\nu^T.^{[3]} \tag{8}$$

For any $k \in \mathbb{N}$, there exists p_k such that $p_k h \leq k < (p_k+1)h$. And since matrix $\Xi(m+l)$ is stochastic,

$$\lim_{k\to\infty}\left(\prod_{l=0}^{k} \Xi(m+l) - \mathbf{1}\nu^T\right) = \lim_{k\to\infty}\left(\prod_{l=p_k h}^{k} \Xi(m+l)\right) \times \left(\prod_{l=0}^{p_k h-1} \Xi(m+l) - \mathbf{1}\nu^T\right).$$

Since $\mathcal{M}$ is compact, we get that matrix $\prod_{l=p_k h}^k \Xi(m+l)$ in the above equation belongs to a bounded set. Furthermore, it follows from Eq. (8) that

$$\lim_{k\to\infty}\left(\prod_{l=0}^{p_k h-1} \Xi(m+l) - \mathbf{1}\nu^T\right) = \lim_{p_k\to\infty} \prod_{l=0}^{p_k h-1} \Xi(m+l) - \mathbf{1}\nu^T = 0.$$

Thus,

$$\lim_{k\to\infty}\left(\prod_{l=0}^{k} \Xi(m+l) - \mathbf{1}\nu^T\right) = 0,$$

which yields that

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty} \prod_{l=0}^{k} \Xi(m+l) y(m) = \mathbf{1}\left(\nu^T y(m)\right),$$

that is, system (7) solves a full state consensus problem. □

6. Conclusion

This paper considered the partial state consensus problem in networks of second-order agents with dynamically changing interaction topologies and time-varying communication time-delays and proposed an asynchronous partial state consensus protocol, whose validity can be guaranteed under the relaxable condition that position-like states as the only information are transmitted intermittently over the network. Nevertheless, there still exist some other interesting problems that need to be addressed, such as the design and analysis of a full state consensus protocol in the framework of this paper.

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