New delay-dependent stability criteria for neural networks with time-varying interval delay ☆

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ABSTRACT

The problem of stability analysis of neural networks with time-varying delay in a given range is investigated in this Letter. By introducing a new Lyapunov functional which uses the information on the lower bound of the delay sufficiently and an augmented Lyapunov functional which contains some triple-integral terms, some improved delay-dependent stability criteria are derived using the free-weighting matrices method. Numerical examples are presented to illustrate the less conservatism of the obtained results and the effectiveness of the proposed method.

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1. Introduction

Neural networks have been applied in many areas such as pattern recognition, data mining, signal filtering, financial prediction and adaptive control. Since there inevitably exist integration and communication delay, stability of the delayed neural network has been extensively studied. Existing stability criteria can be classified into two categories, namely, delay-independent ones [1–4] and delay-dependent ones [5–21]. Since delay-independent stability criteria are usually conservative than delay-dependent ones especially when the delay is small, delay-dependent stability criteria have received much attention.

By introducing a new Lyapunov functional and using the S-procedure, a less conservative stability condition was put forward in [8]. In order to avoid the conservatism involve by model transformation and bounding techniques for cross terms, free-weighting matrices method was used to derive stability criteria for neural networks with time-varying delay [9]. Results in [9] were further improved in [10] by considering some useful terms which were ignored in previous results when estimating the upper bound on the derivative of the Lyapunov functional. Using Jensen's inequality, some simplified stability criteria were proposed [22]. These criteria were equivalent to those in [10] but with less decision variables. By constructing an augmented Lyapunov functional, improved stability conditions have been established in [23]. Using the relationship that $d(t) + (h_2 - d(t)) = h_2$ and $(d(t) - h_1) + (h_2 - d(t)) = h_2 - h_1$, some improved delay-dependent stability criteria were proposed in [11]. However, the above results are still conservative to some extent and there exists room for further improvement.

In practice, the lower bound of the delay is not always 0. Therefore, the delay considered in this Letter is assumed to belong to a given interval. By introducing a new Lyapunov functional, less conservative results are obtained using the free-weighting matrices method and the idea of convex combination [16]. Using the augmented Lyapunov functional approach, the obtained results are further improved. Two numerical examples are given to show the effectiveness of the proposed method and the less conservatism of the obtained results.

2. Problem formulation and preliminaries

Consider the following neural network with time-varying interval delay:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - d(t))) + u \tag{1}$$

where $x(\cdot) = [x_1(\cdot)x_2(\cdot) \cdots x_n(\cdot)]^T$ is the neuron state vector, $g(x(\cdot)) = [g_1(x_1(\cdot))g_2(x_2(\cdot)) \cdots g_n(x_n(\cdot))]^T$ is the neuron activation function, and $u = [u_1 u_2 \cdots u_n]^T$ is a constant input vector. $C = \text{diag}\{c_1, c_2, \ldots, c_n\}$ with $c_i > 0$, $i = 1, 2, \ldots, n$, is a diagonal matrix representing self-feedback term, A is the connection weight matrix and B is the delayed connection weight matrix. The delay $d(t)$ is a time-varying differentiable function satisfying

$$h_1 \leqslant d(t) \leqslant h_2 \tag{2}$$

and

$$\dot{d}(t) \leqslant \mu \tag{3}$$

where $h_2 > h_1 > 0$, $\mu \geqslant 0$ are constants. It is assumed that each neuron activation function, $g_i(\cdot)$, $i = 1, 2, \ldots, n$, is nondecreasing, bounded and satisfying the following condition:

$$0 \leqslant \frac{g_i(x) - g_i(y)}{x - y} \leqslant k_i \quad \forall x, y \in \mathbb{R}, \ x \neq y, \ i = 1, 2, \ldots, n \tag{4}$$

where k_i, $i = 1, 2, \ldots, n$ are positive constants.

Assuming that $x^* = [x_1^* x_2^* \cdots x_n^*]$ is the equilibrium point of (1) and using the transformation $z(\cdot) = x(\cdot) - x^*$, (1) can be converted to the following error system:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t - \tau(t))) \tag{5}$$

where $z(\cdot) = [z_1(\cdot)z_2(\cdot) \cdots z_n(\cdot)]^T$ is the state vector, $f(z(\cdot)) = [f_1(z_1(\cdot))f_2(z_2(\cdot)) \cdots f_n(z_n(\cdot))]^T$, and $f_i(z_i(\cdot)) = g_i(z_i(\cdot) + x_i^*) - g_i(x_i^*)$, $i = 1, 2, \ldots, n$.

According to (4), one can obtained that the functions $f_i(\cdot)$, $i = 1, 2, \ldots, n$, satisfy the following condition:

$$0 \leqslant \frac{f_i(z_i)}{z_i} \leqslant k_i, \quad f_i(0) = 0, \ \forall z_i \neq 0, \ i = 1, 2, \ldots, n \tag{6}$$

which is equivalent to

$$f_i(z_i)[f_i(z_i) - k_i z_i] \leqslant 0, \quad f_i(0) = 0, \ i = 1, 2, \ldots, n. \tag{7}$$

3. Main results

In this section, some new Lyapunov functionals are introduced and less conservative delay-dependent stability criteria are derived for system (5) with time-varying delay satisfying (2)–(3).

3.1. New stability results

In previous works such as [11], the Lyapunov functional which uses the information on both the upper bound of the delay and the lower bound of the delay is often of the following form:

$$\begin{aligned}
\bar{V}(z_t) = {}& z^T(t)Pz(t) + 2\sum_{i=1}^{n} \lambda_i \int_0^{z_i} f_i(s)\,ds + \int_{t-d(t)}^{t} z^T(s)Q_1 z(s) + \int_{t-d(t)}^{t} f^T(z(s))Q_2 f(z(s)) + \int_{t-h_1}^{t} z^T(s)Q_3 z(s) \\
& + \int_{t-h_2}^{t} z^T(s)Q_4 z(s) + \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds\,d\theta + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{z}^T(s)Z_2 \dot{z}(s)\,ds\,d\theta.
\end{aligned} \tag{8}$$

Although the lower bound of the delay, h_1, is used in the above Lyapunov functional, we think that the lower bound of the delay, h_1, is not used sufficiently especially when h_1 is not zero. In the above Lyapunov functional, the upper limit of some integral terms should be $t - h_1$ but not t and the lower limit of the outer integral of a double integral term should be h_1 but not h_2. In this Letter, a new kind of Lyapunov functional being of the following form is proposed

$$\begin{aligned}
V(z_t) = {}& z^T(t)Pz(t) + 2\sum_{i=1}^{n} \lambda_i \int_0^{z_i} f_i(s)\,ds + \int_{t-d(t)}^{t-h_1} z^T(s)Q_1 z(s) + \int_{t-d(t)}^{t} f^T(z(s))Q_2 f(z(s)) + \int_{t-h_1}^{t} z^T(s)Q_3 z(s) \\
& + \int_{t-h_2}^{t-h_1} z^T(s)Q_4 z(s) + \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds\,d\theta + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{z}^T(s)Z_2 \dot{z}(s)\,ds\,d\theta.
\end{aligned} \tag{9}$$

Based on the Lyapunov functional (9), the following theorem presents a delay-dependent stability condition for system (5).

Theorem 1. *For given scalars $h_2 > h_1 > 0$ and $\mu \geqslant 0$, system (5) is asymptotically stable for any time-varying delay satisfying (2)–(3) if there exist matrices $P > 0$, $Q_j > 0$, $j = 1, 2, 3, 4$, $Z_1 > 0$, $Z_2 > 0$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geqslant 0$, and $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \ldots, W_{ni}\} \geqslant 0$, $i = 1, 2$, and any matrices $N_1, N_2, M_1, M_2, S_1,$ and S_2 with appropriate dimensions such that the following LMIs hold:*

$$\Phi_1 = \begin{bmatrix} \Phi & \Gamma Y & h_1 N & h_{12} S \\ * & -Y & 0 & 0 \\ * & * & -h_1 Z_1 & 0 \\ * & * & * & -h_{12} Z_2 \end{bmatrix} < 0, \tag{10}$$

$$\Phi_2 = \begin{bmatrix} \Phi & \Gamma Y & h_1 N & h_{12} M \\ * & -Y & 0 & 0 \\ * & * & -h_1 Z_1 & 0 \\ * & * & * & -h_{12} Z_2 \end{bmatrix} < 0 \tag{11}$$

where

$$\Phi_{11} = -PC - C^T P + Q_3 + N_1 + N_1^T, \qquad \Phi_{12} = N_2^T + M_1 - S_1, \qquad \Phi_{13} = PA - C^T \Lambda + K W_1,$$

$$\Phi_{22} = -(1-\mu)Q_1 - S_2 - S_2^T + M_2 + M_2^T, \qquad \Phi_{33} = Q_2 - 2W_1 + \Lambda A + A^T \Lambda^T, \qquad \Phi_{44} = -(1-\mu)Q_2 - 2W_2,$$

$$\Phi_{55} = -Q_3 + Q_1 + Q_4, \qquad Y = h_1 Z_1 + h_{12} Z_2, \qquad \Gamma = [-C\ 0\ A\ B\ 0\ 0]^T, \qquad N = [N_1^T\ N_2^T\ 0\ 0\ 0\ 0]^T,$$

$$S = [S_1^T\ S_2^T\ 0\ 0\ 0\ 0]^T, \qquad M = [M_1^T\ M_2^T\ 0\ 0\ 0\ 0]^T, \qquad K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}, \qquad h_{12} = h_2 - h_1,$$

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & PB & S_1 - N_1 & -M_1 \\ * & \Phi_{22} & 0 & KW_2 & S_2 - N_2 & -M_2 \\ * & * & \Phi_{33} & \Lambda B & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & -Q_4 \end{bmatrix}.$$

Proof. Taking the derivative of $V(z_t)$ along the trajectories of system (5) yields

$$\begin{aligned} \dot{V}(z_t) &= 2z^T(t)P\dot{z}(t) + 2\sum_{i=1}^{n}\lambda_i f_i(z_i(t))\dot{z}_i(t) - (1-\dot{d}(t))z^T(t-d(t))Q_1 z(t-d(t)) + f^T(z(t))Q_2 f(z(t)) \\ &\quad - (1-\dot{d}(t))f^T(z(t-d(t)))Q_2 f(z(t-d(t))) + z^T(t)Q_3 z(t) - z^T(t-h_1)(Q_3 - Q_1 - Q_4)z(t-h_1) \\ &\quad - z^T(t-h_2)Q_4 z(t-h_2) + h_1 \dot{z}^T(t)Z_1 \dot{z}(t) - \int_{t-h_1}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds + h_{12}\dot{z}^T(t)Z_2\dot{z}(t) - \int_{t-h_2}^{t-h_1} \dot{z}^T(s)Z_2\dot{z}(s)\,ds \\ \end{aligned} \tag{12}$$

$$\begin{aligned} &\leqslant 2z^T(t)P\dot{z}(t) + 2f^T(z(t))\Lambda\dot{z}(t) - (1-\mu)z^T(t-d(t))Q_1 z(t-d(t)) + f^T(z(t))Q_2 f(z(t)) \\ &\quad - (1-\mu)f^T(z(t-d(t)))Q_2 f(z(t-d(t))) + z^T(t)Q_3 z(t) - z^T(t-h_1)(Q_3 - Q_1 - Q_4)z(t-h_1) \\ &\quad - z^T(t-h_2)Q_4 z(t-h_2) + h_1 \dot{z}^T(t)Z_1 \dot{z}(t) - \int_{t-h_1}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds \\ &\quad + h_{12}\dot{z}^T(t)Z_2\dot{z}(t) - \int_{t-d(t)}^{t-h_1} \dot{z}^T(s)Z_2\dot{z}(s)\,ds - \int_{t-h_2}^{t-d(t)} \dot{z}^T(s)Z_2\dot{z}(s)\,ds. \end{aligned} \tag{13}$$

Similar to [11], the following equalities hold

$$0 = 2\xi^T(t)N\left[z(t) - z(t-h_1) - \int_{t-h_1}^{t} \dot{z}(s)\,ds\right], \tag{14}$$

$$0 = 2\xi^T(t)S\left[z(t-h_1) - z(t-d(t)) \int_{t-d(t)}^{t-h_1} \dot{z}(s)ds\right] \tag{15}$$

and,

$$0 = 2\xi^T(t) M \left[z(t - d(t)) - z(t - h_2) - \int_{t-h_2}^{t-d(t)} \dot{z}(s)\, ds \right] \tag{16}$$

where $\xi(t) = [z^T(t) \ z^T(t - d(t)) \ f^T(z(t)) \ f^T(z(t - d(t))) \ z^T(t - h_1) \ z^T(t - h_2)]^T$. It is easy to see that

$$-2\xi^T(t) N \int_{t-h_1}^{t} \dot{z}(s)\, ds \leqslant h_1 \xi^T(t) N Z_1^{-1} N^T \xi(t) + \int_{t-h_1}^{t} \dot{z}^T(s) Z_1 \dot{z}(s), \tag{17}$$

$$-2\xi^T(t) S \int_{t-d(t)}^{t-h_1} \dot{z}(s)\, ds \leqslant (d(t) - h_1) \xi^T(t) S Z_2^{-1} S^T \xi(t) + \int_{t-d(t)}^{t-h_1} \dot{z}^T(s) Z_2 \dot{z}(s) \tag{18}$$

and

$$-2\xi^T(t) M \int_{t-h_2}^{t-d(t)} \dot{z}(s)\, ds \leqslant (h_2 - d(t)) \xi^T(t) M Z_2^{-1} M^T \xi(t) + \int_{t-h_2}^{t-d(t)} \dot{z}^T(s) Z_2 \dot{z}(s). \tag{19}$$

From (7), it can be obtained that

$$f_i(z_i(t)) \big[f_i(z_i(t)) - k_i z_i(t) \big] \leqslant 0, \quad i = 1, 2, \dots, n \tag{20}$$

and

$$f_i(z_i(t - \tau(t))) \big[f_i(z_i(t - \tau(t))) - k_i z_i(t - \tau(t)) \big] \leqslant 0, \quad i = 1, 2, \dots, n. \tag{21}$$

Clearly, the following inequality holds for any $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \dots, W_{ni}\} \geqslant 0$, $i = 1, 2$.

$$0 \leqslant -2 \sum_{i=1}^{n} W_{i1} f_i(z_i(t)) \big[f_i(z_i(t)) - k_i z_i(t) \big] - 2 \sum_{i=1}^{n} W_{i2} f_i(z_i(t - \tau(t))) \big[f_i(z_i(t - \tau(t))) - k_i z_i(t - \tau(t)) \big]$$

$$= 2z^T(t) K W_1 f(z(t)) - 2 f^T(z(t)) W_1 f(z(t)) + 2 z^T(t - \tau(t)) K W_2 f(z(t - \tau(t))) - 2 f^T(z(t - \tau(t))) W_2 f(z(t - \tau(t))). \tag{22}$$

Add the both sides of (14)–(16) and (22) to both sides of (13) and apply (17)–(19), and one can obtain that

$$\dot{V}(z_t) \leqslant \xi^T(t) \Sigma \xi(t) \tag{23}$$

where

$$\Sigma = \Phi + \Gamma Y \Gamma^T + h_1 N Z_1^{-1} N^T + (d(t) - h_1) S Z_2^{-1} S^T + (h_2 - d(t)) M Z_2^{-1} M^T.$$

Note that $h_1 \leqslant d(t) \leqslant h_2$, $(d(t) - h_1) S Z_2^{-1} S^T + (h_2 - d(t)) M Z_2^{-1} M^T$ can be seen as the convex combination of $S Z_2^{-1} S^T$ and $M Z_2^{-1} M^T$ on $d(t)$. Therefore, $\Sigma < 0$ holds if and only if

$$\Phi + \Gamma Y \Gamma^T + h_1 N Z_1^{-1} N^T + h_{12} S Z_2^{-1} S^T < 0 \tag{24}$$

and

$$\Phi + \Gamma Y \Gamma^T + h_1 N Z_1^{-1} N^T + h_{12} M Z_2^{-1} M^T < 0. \tag{25}$$

By Schur complements, (24) is equivalent to $\Phi_1 < 0$ and (25) is equivalent to $\Phi_2 < 0$. Therefore, if $\Phi_1 < 0$ and $\Phi_2 < 0$, then $\dot{V}(z_t) < -\varepsilon \|z(t)\|^2$ for a sufficiently small $\varepsilon > 0$ which means that system (5) is asymptotically stable according to the Lyapunov stability theory [25]. □

Remark 2. A new Lyapunov functional (9) is introduced where the information on the lower bound of the delay is used sufficiently. Based on this Lyapunov functional, a new stability condition is developed, which will be illustrated to be less conservative than existing results through some numerical examples.

When μ is unknown, Theorem 1 is inapplicable any more. For this case, the following rate-independent result can be obtained by letting $Q_1 = \epsilon_1 I$ and $Q_2 = \epsilon_2 I$ with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ being sufficient small scalars.

Corollary 3. *For given scalars $h_2 > h_1 > 0$, system (5) is asymptotically stable for any time-varying delay satisfying (2) if there exist matrices $P > 0$, $Q_j > 0$, $j = 3, 4$, $Z_1 > 0$, $Z_2 > 0$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} \geqslant 0$, and $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \dots, W_{ni}\} \geqslant 0$, $i = 1, 2$, and any matrices N_1, N_2, M_1, M_2, S_1, and S_2 with appropriate dimensions such that the following LMIs hold:*

$$\hat{\Phi}_1 = \begin{bmatrix} \hat{\Phi} & \Gamma Y & h_1 N & h_{12} S \\ * & -Y & 0 & 0 \\ * & * & -h_1 Z_1 & 0 \\ * & * & * & -h_{12} Z_2 \end{bmatrix} < 0, \tag{26}$$

$$\hat{\Phi}_2 = \begin{bmatrix} \hat{\Phi} & \Gamma Y & h_1 N & h_{12} M \\ * & -Y & 0 & 0 \\ * & * & -h_1 Z_1 & 0 \\ * & * & * & -h_{12} Z_2 \end{bmatrix} < 0 \tag{27}$$

where

$$\hat{\Phi} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & PB & S_1 - N_1 & -M_1 \\ * & \hat{\Phi}_{22} & 0 & KW_2 & S_2 - N_2 & -M_2 \\ * & * & \hat{\Phi}_{33} & \Lambda B & 0 & 0 \\ * & * & * & \hat{\Phi}_{44} & 0 & 0 \\ * & * & * & * & \hat{\Phi}_{55} & 0 \\ * & * & * & * & * & -Q_4 \end{bmatrix},$$

$$\hat{\Phi}_{22} = -S_2 - S_2^T + M_2 + M_2^T, \qquad \hat{\Phi}_{33} = -2W_1 + \Lambda A + A^T \Lambda^T, \qquad \hat{\Phi}_{44} = -2W_2, \qquad \hat{\Phi}_{55} = -Q_3 + Q_4$$

and the other terms are the same as defined in Theorem 1.

3.2. Further improvements

In this part, the results presented in the above are further improved by introducing an augmented Lyapunov functional which contains some novel triple-integral terms.

The following augmented Lyapunov functional is introduced

$$V_a(z_t) = \eta^T(t) P \eta(t) + 2 \sum_{i=1}^n \lambda_i \int_0^{z_i} f_i(s)\,ds + \int_{t-d(t)}^{t-h_1} z^T(s) Q_1 z(s) + \int_{t-d(t)}^{t} f^T(z(s)) Q_2 f(z(s)) + \int_{t-h_1}^{t} z^T(s) Q_3 z(s)$$

$$+ \int_{t-h_2}^{t-h_1} z^T(s) Q_4 z(s) + \int_{t-h_1}^{t} \dot{z}^T(s) Q_5 \dot{z}(s) + \int_{t-h_2}^{t-h_1} \dot{z}^T(s) Q_6 \dot{z}(s) + \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}^T(s) Z_1 \dot{z}(s)\,ds\,d\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}^T(s) Z_2 \dot{z}(s)\,ds\,d\theta$$

$$+ \int_{-h_1}^{0} \int_{t+\theta}^{t} z^T(s) Z_3 z(s)\,ds\,d\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} z^T(s) Z_4 z(s)\,ds\,d\theta + \int_{-h_1}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s) R_1 \dot{z}(s)\,ds\,d\lambda\,d\theta$$

$$+ \int_{-h_2}^{-h_1} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s) R_2 \dot{z}(s)\,ds\,d\lambda\,d\theta \tag{28}$$

where $\eta^T(t) = [z^T(t)\ z^T(t-h_1)\ z^T(t-h_2)\ \int_{t-h_1}^{t} z^T(s)\,ds\ \int_{t-h_2}^{t-h_1} z^T(s)\,ds]$.

Based on the above augmented Lyapunov functional (28), we have the following result.

Theorem 4. *For given scalars $h_2 > h_1 > 0$ and $\mu \geqslant 0$, system (5) is asymptotically stable for any time-varying delay satisfying (2) and (3) if there exist matrices $P = [P_{ij}]_{5 \times 5} > 0$, $Q_j > 0$, $j = 1, \ldots, 6$, $Z_l > 0$, $l = 1, \ldots, 4$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geqslant 0$, and $W_i = \mathrm{diag}\{W_{1i}, W_{2i}, \ldots, W_{ni}\} \geqslant 0$, $i = 1, 2$, and any matrices $N_1, N_2, M_1, M_2, S_1,$ and S_2 with appropriate dimensions such that the following LMIs hold:*

$$\Theta_1 = \begin{bmatrix} \Theta & \Gamma Y & \frac{h_1^2}{2} L & h_\delta H & h_1 N & h_1 \Upsilon_1 & h_{12} S & h_{12} \Upsilon_2 \\ * & -Y & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\frac{h_1^2}{2} R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_\delta R_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -h_1 Z_1 & 0 & 0 & 0 \\ * & * & * & * & * & -h_1 Z_3 & 0 & 0 \\ * & * & * & * & * & * & -h_{12} Z_2 & 0 \\ * & * & * & * & * & * & * & -h_{12} Z_4 \end{bmatrix} < 0, \tag{29}$$

$$
\Theta_2 = \begin{bmatrix}
\Theta & \Gamma Y & \frac{h_1^2}{2}L & h_\delta H & h_1 N & h_1 \Upsilon_1 & h_{12} M & h_{12} \Upsilon_2 \\
* & -Y & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\frac{h_1^2}{2}R_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h_\delta R_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_1 Z_1 & 0 & 0 & 0 \\
* & * & * & * & * & -h_1 Z_3 & 0 & 0 \\
* & * & * & * & * & * & -h_{12} Z_2 & 0 \\
* & * & * & * & * & * & * & -h_{12} Z_4
\end{bmatrix} < 0 \tag{30}
$$

where

$$
\Theta = \begin{bmatrix}
\Theta_{11} & \Theta_{12} & \Theta_{13} & P_{11}B & \Theta_{15} & \Theta_{16} & P_{12} & P_{13} \\
* & \Theta_{22} & 0 & KW_2 & S_2 - N_2 & -M_2 & 0 & 0 \\
* & * & \Theta_{33} & \Lambda B & A^T P_{12} & A^T P_{13} & 0 & 0 \\
* & * & * & \Theta_{44} & B^T P_{12} & B^T P_{13} & 0 & 0 \\
* & * & * & * & \Theta_{55} & \Theta_{56} & P_{22} & P_{23} \\
* & * & * & * & * & \Theta_{66} & P_{23}^T & P_{33} \\
* & * & * & * & * & * & Q_6 - Q_5 & 0 \\
* & * & * & * & * & * & * & -Q_6
\end{bmatrix},
$$

$\Theta_{11} = -P_{11}C - C^T P_{11} + Q_3 + h_1 Z_3 + h_{12} Z_4 + N_1 + N_1^T + h_1 L1 + h_1 L_1^T + h_{12} H_1 + h_{12} H_1^T,$

$\Theta_{12} = N_2^T + M_1 - S_1 + h_1 L_2^T + h_{12} H_2^T, \qquad \Theta_{13} = P_{11}A - C^T \Lambda + K W_1, \qquad \Theta_{15} = -C^T P_{12} - P_{14} + P_{15} + P_{24}^T - N_1 + S_1,$

$\Theta_{16} = -C^T P_{13} - P_{15} + P_{34}^T - M_1, \qquad \Theta_{22} = -(1-\mu)Q_1 - S_2 - S_2^T + M_2 + M_2^T, \qquad \Theta_{33} = Q_2 - 2W_1 + \Lambda A + A^T \Lambda^T,$

$\Theta_{44} = -(1-\mu)Q_2 - 2W_2, \qquad \Theta_{55} = -Q_3 + Q_1 + Q_4 - P_{24} - P_{24}^T + P_{25}^T + P_{25}^T, \qquad \Theta_{56} = -P_{25} - P_{34}^T + P_{35}^T,$

$\Theta_{66} = -Q_4 - P_{35} - P_{35}^T, \qquad Y = Q_5 + h_1 Z_1 + h_{12} Z_2 + \frac{h_1^2}{2}R_1 + h_\delta R_2, \qquad \Gamma = [-C\ 0\ A\ B\ 0\ 0\ 0\ 0]^T,$

$N = [N_1^T\ N_2^T\ 0\ 0\ 0\ 0\ 0\ 0]^T, \qquad S = [S_1^T\ S_2^T\ 0\ 0\ 0\ 0\ 0\ 0]^T, \qquad M = [M_1^T\ M_2^T\ 0\ 0\ 0\ 0\ 0\ 0]^T,$

$\Upsilon_1 = [P_{14}^T C - P_{44} - L_1^T\ L_2^T\ -P_{14}^T A\ -P_{14}^T B\ P_{44}\ P_{45}\ -P_{24}^T\ -P_{34}^T],$

$\Upsilon_2 = [P_{15}^T C - P_{45} - H_1^T\ H_2^T\ -P_{15}^T A\ -P_{15}^T B\ P_{55}\ -P_{45}\ P_{55}\ -P_{25}^T\ -P_{35}^T],$

$K = \text{diag}\{k_1, k_2, \ldots, k_n\}, \qquad h_{12} = h_2 - h_1, \qquad h_\delta = (h_2^2 - h_1^2)/2.$

Proof. Taking the derivative of $V_a(z_t)$ described by (28) along the trajectories of system (5) yields

$$
\begin{aligned}
\dot{V}_a(z_t) = {}& 2\eta^T(t)P\dot{\eta}(t) + 2\sum_{i=1}^{n}\lambda_i f_i(z_i(t))\dot{z}_i(t) - (1-\dot{d}(t))z^T(t-d(t))Q_1 z(t-d(t)) + f^T(z(t))Q_2 f(z(t)) \\
& - (1-\dot{d}(t))f^T(z(t-d(t)))Q_2 f(z(t-d(t))) + z^T(t)Q_3 z(t) - z^T(t-h_1)(Q_3 - Q_1 - Q_4)z(t-h_1) \\
& - z^T(t-h_2)Q_4 z(t-h_2) - \dot{z}^T(t-h_1)(Q_5 - Q_6)\dot{z}(t-h_1) + \dot{z}^T(t)Q_5 \dot{z}(t) - \dot{z}^T(t-h_2)Q_6 \dot{z}(t-h_2) \\
& + h_1 \dot{z}^T(t)Z_1 \dot{z}(t) - \int_{t-h_1}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds + h_{12}\dot{z}^T(t)Z_2 \dot{z}(t) - \int_{t-h_2}^{t-h_1} \dot{z}^T(s)Z_2 \dot{z}(s)\,ds \\
& + h_1 z^T(t)Z_3 z(t) - \int_{t-h_1}^{t} z^T(s)Z_3 z(s)\,ds + h_{12}z^T(t)Z_2 z(t) - \int_{t-h_2}^{t-h_1} z^T(s)Z_4 z(s)\,ds \\
& + \frac{1}{2}h_1^2 \dot{z}^T(t)R_1 \dot{z}(t) - \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)R_1 \dot{z}(s)\,ds\,d\theta + h_\delta \dot{z}^T(t)R_2 \dot{z}(t) - \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{z}^T(s)R_2 \dot{z}(s)\,ds\,d\theta \\
\leqslant {}& 2\eta^T(t)P\dot{\eta}(t) + 2f^T(z(t))\Lambda\dot{z}(t) - (1-\mu)z^T(t-d(t))Q_1 z(t-d(t)) + f^T(z(t))Q_2 f(z(t)) \\
& - (1-\mu)f^T(z(t-d(t)))Q_2 f(z(t-d(t))) + z^T(t)Q_3 z(t) - z^T(t-h_1)(Q_3 - Q_1 - Q_4)z(t-h_1) \\
& - z^T(t-h_2)Q_4 z(t-h_2) - \dot{z}^T(t-h_1)(Q_5 - Q_6)\dot{z}(t-h_1) + \dot{z}^T(t)Q_5 \dot{z}(t) - \dot{z}^T(t-h_2)Q_6 \dot{z}(t-h_2) \\
& + h_1 \dot{z}^T(t)Z_1 \dot{z}(t) - \int_{t-h_1}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)\,ds + h_{12}\dot{z}^T(t)Z_2 \dot{z}(t) - \int_{t-d(t)}^{t-h_1} \dot{z}^T(s)Z_2 \dot{z}(s)\,ds
\end{aligned} \tag{31}
$$

$$- \int_{t-h_2}^{t-d(t)} \dot{z}^T(s) Z_2 \dot{z}(s) \, ds + h_1 z^T(t) Z_3 z(t) - \int_{t-h_1}^{t} z^T(s) Z_3 z(s) \, ds + h_{12} z^T(t) Z_4 z(t)$$

$$- \int_{t-h_2}^{t-h_1} z^T(s) Z_2 z(s) \, ds + \frac{1}{2} h_1^2 \dot{z}^T(t) R_1 \dot{z}(t) - \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}^T(s) R_1 \dot{z}(s) \, ds \, d\theta$$

$$+ h_\delta \dot{z}^T(t) R_2 \dot{z}(t) - \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}^T(s) R_2 \dot{z}(s) \, ds \, d\theta. \tag{32}$$

To deal with the double-integral terms $-\int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}^T(s) R_1 \dot{z}(s) \, ds \, d\theta$ and $-\int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}^T(s) R_2 \dot{z}(s) \, ds \, d\theta$ in (32), the following two equations are introduced

$$0 = 2\xi^T(t) L \left[h_1 z(t) - \int_{t-h_1}^{t} z(s) \, ds - \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}(s) \, ds \, d\theta \right] \tag{33}$$

and

$$0 = 2\xi^T(t) H \left[h_{12} z(t) - \int_{t-h_2}^{t-d(t)} z(s) \, ds - \int_{t-d(t)}^{t-h_1} z(s) \, ds - \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}(s) \, ds \, d\theta \right]. \tag{34}$$

In addition, the following two inequalities hold

$$-2\xi^T(t) L \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}(s) \, ds \, d\theta \leqslant \frac{1}{2} h_1^2 \xi^T(t) L R_1^{-1} L^T \xi(t) + \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{z}(s) R_1 \dot{z}(s) \, ds \, d\theta \tag{35}$$

and

$$-2\xi^T(t) H \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}(s) \, ds \, d\theta \leqslant h_\delta \xi^T(t) H R_2^{-1} H^T \xi(t) + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{z}(s) R_2 \dot{z}(s) \, ds \, d\theta. \tag{36}$$

Base on (33)–(36) and following a similar line to Theorem 1, the proof can be completed. □

Remark 5. Similarly, Theorem 4 can also be extended to deal with the case for unknown μ. Due to the limitation of the space, it is omitted here.

Remark 6. Recently, a delay decomposition approach has been proposed in [26]. This scheme is very effective in the reduction of the conservatism. Combine the Lyapunov functional proposed in this Letter with the delay decomposition approach, and further less conservative results can be obtained. Due to the limitation of the space, the details are omitted here.

4. Numerical examples

In this section, two numerical examples are presented to show the less conservatism of our results and the effectiveness of the proposed method.

Example 1. Consider the following delayed neural network with [11,18]

$$C = \text{diag}\{1.2769, 0.6231, 0.9230, 0.4480\},$$

$$A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9634 & -0.5015 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$$k_1 = 0.1137, \quad k_2 = 0.1279, \quad k_3 = 0.7994, \quad k_4 = 0.2368.$$

Table 1
Upper bounds on h_2 for various h_1 and μ.

h_1	Methods	$\mu = 0.5$	$\mu = 0.9$	Unknown μ
1	[11]	2.5802	2.2736	2.2393
	[27]	1.8832	1.7657	1.7651
	[28]	2.2958	1.9512	1.9224
	Theorem 1	2.5848	2.3111	2.2770
	Theorem 4	2.6869	2.3924	2.3540
2	[11]	2.7500	2.6468	2.6299
	[27]	2.4340	2.4003	2.4001
	[28]	2.5778	2.4849	2.4712
	Theorem 1	2.7716	2.6670	2.6504
	Theorem 4	2.8475	2.7375	2.7190
3	[11]	3.1733	3.1155	3.1042
	[27]	3.0956	3.0682	3.0671
	[28]	3.1321	3.0872	3.0786
	Theorem 1	3.1772	3.1186	3.1072
	Theorem 4	3.2429	3.1827	3.1711

Table 2
Upper bounds on h_2 for various h_1 and μ.

h_1	Methods	$\mu = 0.8$	$\mu = 0.9$	Unknown μ
0.5	[11]	1.3566	1.1689	1.0263
	[27]	0.8262	0.8215	0.8183
	[28]	1.1217	0.9984	0.9037
	Theorem 1	1.3599	1.1786	1.0391
	Theorem 4	1.3600	1.1786	1.0437
0.75	[11]	1.3856	1.2110	1.0803
	[27]	0.9669	0.9625	0.9592
	[28]	1.2213	1.1021	1.0102
	Theorem 1	1.3990	1.2240	1.0885
	Theorem 4	1.3990	1.2241	1.0972
1	[11]	1.4578	1.2887	1.1641
	[27]	1.1152	1.1108	1.1075
	[28]	1.3432	1.2238	1.1318
	Theorem 1	1.4692	1.2944	1.1656
	Theorem 4	1.4692	1.2948	1.1774

It is assumed that $\dot{d}(t) \leqslant \mu$. The corresponding upper bounds on h_2 for various μ and h_1 calculated by Theorem 1 are listed in Table 1 compared with those in [11,27,28]. Table 1 also lists the results for unknown μ. It can be seen that results obtained by Theorem 1 in this Letter are less conservative than those in [11,27,28] since a new Lyapunov functional which sufficiently uses the information on the lower bound of the delay is introduced in the development of Theorem 1. It can also be seen that results obtained by Theorem 4 are less conservative than those obtained by Theorem 1 because some triple-integral terms are introduced in the Lyapunov functional in the derivation of the Theorem 4.

Theorem 4 is checked on an Intel Core (TM) 2Duo® processor at 2.20 GHz using Matlab LMI toolbox. The computation time for this example is about 150.3 s.

Example 2. Consider the following delayed neural network with [24]

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 0.5 \\ 0.5 & -2 \end{bmatrix}, \qquad k_1 = 0.4, \qquad k_2 = 0.8.$$

The objective is to compute the upper bound of h_2 for various h_1 and μ. The compared results are listed in Table 2. It can be seen that the method proposed in this Letter yields less conservative results than those in the literature. For this example, the computation time of Theorem 4 is about 9.7 s.

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