Published in IET Control Theory and Applications
Received on 2nd November 2008
Revised on 30th May 2009
doi: 10.1049/iet-cta.2008.0508

ISSN 1751-8644

Brief Paper

Improved stability criteria for linear systems with time-varying delay

J. Sun[1,2] G.P. Liu[2,3] J. Chen[1] D. Rees[2]

[1]School of Automation, Beijing Institute of Technology, Beijing 100081, People's Republic of China
[2]Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, UK
[3]CTGT Center in Harbin Institute of Technology, Harbin 150001, People's Republic of China
E-mail: helios1225@yahoo.com.cn

Abstract: This study is concerned with the stability analysis of systems with time-varying delay in a given interval. A new type of augmented Lyapunov functional which contains some triple-integral terms is proposed. By introducing free-weighting matrices, a new delay-range-dependent stability criterion is derived in terms of linear matrix inequality. The rate-range of the delay is considered, so the stability criterion is also delay-rate-range dependent. Numerical examples are given to illustrate the effectiveness of the proposed method.

1 Introduction

Time delay is encountered in many dynamic systems such as chemical or process control systems and networked control systems and is often the cause of instability and poor performance [1–3]. The subject of stability analysis of systems with time-varying delay has attracted considerable attention during the past few years [4–9]. Since it is assumed that the delay is unbounded, delay-independent criteria are usually more conservative than delay dependent ones especially when the time-delay is small. Therefore much attention has been paid to the study of delay-dependent stability [10–18].

Most of existing delay-dependent stability criteria are obtained using Lyapunov–Krasovskii approach or Lyapunov–Razumikhin approach combined with model transformations and bounding techniques for cross terms. Among these results, the descriptor model transformation method [19, 20] combined with Moon *et al.*'s inequality [21] was the most efficient. In order to reduce the conservativeness introduced by model transformation and bounding techniques, a free-weighting matrices method was introduced in [22] to derive delay dependent stability criteria. A parameterised model transformation method was combined with the free-weighting matrices method to derive new delay-dependent stability criteria in [23]. These results were further improved in [24] using the augmented Lyapunov functional method. However, only a constant delay was considered in [24]. Recently, some less conservative results [25] have been obtained by considering some useful terms when estimating the upper bound on the derivative of the Lyapunov functional. And, these results were further improved using the idea of constructing an augmented Lyapunov functional in [26]. It should be noted that the Lyapunov functional introduced in the above literature only contain some integral terms and double-integral terms, for example, $\int_{t-\tau_2}^{t} x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s$ and $\int_{-\tau_2}^{0}\int_{t+\beta}^{t}\dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)\mathrm{d}s\mathrm{d}\beta$. In our recent work [27], a new type of Lyapunov functional containing a triple-integral term has been introduced to derive delay-dependent stability conditions for neutral time-delay systems. However, only a constant delay case has been considered in [27]. In this paper, the method in [27] is extended to deal with systems with time-varying delays and less conservative stability criteria are derived using the free-weighting matrices method. Furthermore, the delay derivative is often assumed to satisfy $\dot{d}(t) \leq \mu$ in the literature. The lower bound on the delay derivative is not considered in the literature. In this paper, the delay derivative is assumed to belong to a given interval, that is, $\nu \leq \dot{d}(t) \leq \mu < 1$. The lower bound information on the delay derivative is used in the derivation and a delay-rate-range dependent stability condition is obtained. Numerical examples are given to demonstrate the effectiveness of the proposed method.

2 Problem formulation and main results

Consider the following linear system with time-varying delay

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - d(t)), \quad t > 0 \\ x(t) &= \phi(t), \quad t \in [-\tau_2, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector; the initial condition $\phi(t)$ is a continuously differentiable vector-valued function; $A \in \mathcal{R}^{n\times n}$ and $A_1 \in \mathcal{R}^{n\times n}$ are constant system matrices; $d(t)$ is a time-varying differentiable function and satisfies

$$0 \le \tau_1 \le d(t) < \tau_2 \tag{2}$$

$$\nu \le \dot{d}(t) \le \mu < 1 \tag{3}$$

where τ_1, τ_2 , ν and μ are constants. Let $\tau_{12} = \tau_2 - \tau_1$ and $\tau_s = 1/2(\tau_2^2 - \tau_1^2)$. The following theorem presents a sufficient stability condition for system (1).

Theorem 1: Given scalars $0 \le \tau_1 < \tau_2$ and $\nu \le \mu < 1$, system (1) with a time-varying delay satisfying (2) and (3) is asymptotically stable if there exist matrices $U_1 > 0$, $U_2 > 0$, $P = [P_{ij}]_{5\times 5} > 0$, $Q = [Q_{ij}]_{2\times 2} \ge 0$, $Z = [Z_{ij}]_{2\times 2} \ge 0$, $R = [R_{ij}]_{2\times 2} \ge 0$, $W = [W_{ij}]_{2\times 2} \ge 0$, $X = [X_{ij}]_{2\times 2} \ge 0$, $V = [V_{ij}]_{8\times 8} \ge 0$, $F = [F_{ij}]_{8\times 8} \ge 0$, N, Y, S, H, L, M with appropriate dimensions such that the following LMIs hold:

$$\begin{bmatrix} \Sigma & \frac{1}{2}\tau_2^2 L & \tau_s H \\ \star & -\frac{1}{2}\tau_2^2 U_1 & 0 \\ \star & \star & -\tau_s U_2 \end{bmatrix} < 0 \tag{4}$$

$$\Lambda_1 = \begin{bmatrix} V & \Gamma_1 \\ \star & Z \end{bmatrix} \ge 0 \tag{5}$$

$$\Lambda_2 = \begin{bmatrix} F & \Gamma_2 \\ \star & X \end{bmatrix} \ge 0 \tag{6}$$

$$\Lambda_3 = \begin{bmatrix} V + F & \Gamma_3 \\ \star & Z + X \end{bmatrix} \ge 0 \tag{7}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \star & \Sigma_3 \end{bmatrix} + \Upsilon + \Upsilon^{\mathrm{T}} - MA_c - A_c^{\mathrm{T}} M^T + \tau_2 V + \tau_{12} F$$

$$\Sigma_1 = \mathrm{diag}\{\Pi_{11}, \ -R_{11}(1-\mu), \ -W_{11}, \ -Q_{11}\} + \Psi_1 + \Psi_1^{\mathrm{T}}$$

$$\Sigma_2 = \begin{bmatrix} P_{11} + \Pi_{12} & P_{14} & P_{15} & P_{12} \\ P_{14}^{\mathrm{T}} & P_{44} - R_{12} & P_{45} & P_{24}^{\mathrm{T}} \\ P_{15}^{\mathrm{T}} & P_{45}^{\mathrm{T}} & P_{55} - W_{12} & P_{25}^{\mathrm{T}} \\ P_{12}^{\mathrm{T}} & P_{24} & P_{25} & P_{22} - Q_{12} \end{bmatrix}$$

$$\Sigma_3 = \mathrm{diag}\left\{\Pi_{22} + \frac{1}{2}\tau_2^2 U_1 + \tau_s U_2, \ -R_{22}/(1-\nu), \ -W_{22}, \ -Q_{22}\right\}$$

$$\Upsilon = [N + \tau_2 L + \tau_{12} H \quad Y - N - S \quad S \quad -Y \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Pi_{ij} = Q_{ij} + R_{ij} + W_{ij} + \tau_2 Z_{ij} + \tau_{12} X_{ij}, \quad j = 1, 2, \ i \le j$$

$$A_c = \begin{bmatrix} A & A_1 & 0 & 0 & -I & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} -\Psi_2 + L & N \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} H & S \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} -\Psi_2 + L + H & Y \end{bmatrix}$$

$$\Psi_1 = \begin{bmatrix} P_{13}^{\mathrm{T}} & P_{34} & P_{35} & P_{23}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix}$$

$$\Psi_2 = \begin{bmatrix} P_{33} & 0 & 0 & -P_{33} & P_{13}^{\mathrm{T}} & P_{34} & P_{35} & P_{23}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

Proof: Construct a Lyapunov functional as follows

$$\begin{aligned} V(x_t) &= \zeta^{\mathrm{T}}(t) P \zeta(t) + \int_{t-d(t)}^{t} \varrho^{\mathrm{T}}(s) R \varrho(s)\mathrm{d}s \\ &+ \int_{t-\tau_1}^{t} \varrho^{\mathrm{T}}(s) W \varrho(s)\mathrm{d}s + \int_{t-\tau_2}^{t} \varrho^{\mathrm{T}}(s) Q \varrho(s)\mathrm{d}s \\ &+ \int_{-\tau_2}^{0}\int_{t+\beta}^{t} \varrho^{\mathrm{T}}(s) Z \varrho(s)\mathrm{d}s\mathrm{d}\beta + \int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} \varrho^{\mathrm{T}}(s) X \varrho(s)\mathrm{d}s\mathrm{d}\beta \\ &+ \int_{-\tau_2}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) U_1 \dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\beta \\ &+ \int_{-\tau_2}^{-\tau_1}\int_{\beta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) U_2 \dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\beta \end{aligned} \tag{8}$$

where $\zeta(t) = \mathrm{col}\{x(t), x(t-\tau_2), \int_{t-\tau_2}^{t} x(s)\mathrm{d}s, x(t-d(t)), x(t-\tau_1)\}$, $\varrho(s) = \mathrm{col}\{x(s), \dot{x}(s)\}$. It is easy to see that there exist two positive scalars δ_1 and δ_2 such that $\delta_1 \|x(t)\|^2 \le V(x_t) \le \delta_2 \sup_{-\tau \le \theta \le 0}\{\|x(t+\theta)\|^2, \|\dot{x}(t+\theta)\|^2\}$. Similar to [22, 24, 28], the following equalities hold

$$\alpha_1(t) := 2\xi^{\mathrm{T}}(t) N \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)\,\mathrm{d}s\right] = 0 \tag{9}$$

$$\alpha_2(t) := 2\xi^{\mathrm{T}}(t) Y \left[x(t-d(t)) - x(t-\tau_2) - \int_{t-\tau_2}^{t-d(t)} \dot{x}(s)\,\mathrm{d}s\right] = 0 \tag{10}$$

$$\alpha_3(t) := 2\xi^{\mathrm{T}}(t) S \left[x(t-\tau_1) - x(t-d(t)) - \int_{t-d(t)}^{t-\tau_1} \dot{x}(s)\,\mathrm{d}s\right] = 0 \tag{11}$$

$$\alpha_4(t) := 2\xi^{\mathrm{T}}(t) M[\dot{x}(t) - Ax(t) - A_1 x(t-d(t))] = 0 \tag{12}$$

doi: 10.1049/iet-cta.2008.0508

$$\alpha_5(t) := 2\xi^{\mathrm{T}}(t)L\left[\tau_2 x(t) - \int_{t-\tau_2}^{t-d(t)} x(s)\mathrm{d}s - \int_{t-d(t)}^{t} x(s)\,\mathrm{d}s - \int_{-\tau_2}^{0}\int_{t+\beta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\beta\right] = 0 \quad (13)$$

$$\alpha_6(t) := 2\xi^{\mathrm{T}}(t)H\left[\tau_{12} x(t) - \int_{t-\tau_2}^{t-d(t)} x(s)\mathrm{d}s - \int_{t-d(t)}^{t-\tau_1} x(s)\,\mathrm{d}s - \int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\beta\right] = 0 \quad (14)$$

$$\alpha_7(t) := \tau_2\xi^{\mathrm{T}}(t)V\xi(t) - \int_{t-d(t)}^{t} \xi^{\mathrm{T}}(t)V\xi(t)\mathrm{d}s - \int_{t-\tau_2}^{t-d(t)} \xi^{\mathrm{T}}(t)V\xi(t)\mathrm{d}s = 0 \quad (15)$$

$$\alpha_8(t) := \tau_{12}\xi^{\mathrm{T}}(t)F\xi(t) - \int_{t-d(t)}^{t-\tau_1} \xi^{\mathrm{T}}(t)F\xi(t)\mathrm{d}s - \int_{t-\tau_2}^{t-d(t)} \xi^{\mathrm{T}}(t)F\xi(t)\mathrm{d}s = 0 \quad (16)$$

where $\xi(t) = \mathrm{col}\{x(t), x(t-d(t)), x(t-\tau_1), x(t-\tau_2), \dot{x}(t), \dot{x}(t-d(t))(1-\dot{d}(t)), \dot{x}(t-\tau_1), \dot{x}(t-\tau_2)\}$. Taking the time derivative of $V(x_t)$ along the trajectory of (1) yields

$$\begin{aligned}\dot{V}(x_t) &= 2\zeta^{\mathrm{T}}(t)P\dot{\zeta}(t) + \varrho^{\mathrm{T}}(t)(R+Q+W+\tau_2 Z)\varrho(t) \\ &- (1-\dot{d}(t))\varrho^{\mathrm{T}}(t-d(t))R\varrho(t-d(t)) \\ &- \varrho^{\mathrm{T}}(t-\tau_2)Q\varrho(t-\tau_2) - \varrho^{\mathrm{T}}(t-\tau_1)W\varrho(t-\tau_1) \\ &- \int_{t-d(t)}^{t} \varrho^{\mathrm{T}}(s)Z\varrho(s)\mathrm{d}s - \int_{t-d(t)}^{t-\tau_1} \varrho^{\mathrm{T}}(s)X\varrho(s)\mathrm{d}s \\ &- \int_{t-\tau_2}^{t-d(t)} \varrho^{\mathrm{T}}(s)(Z+X)\varrho(s)\mathrm{d}s + \tau_{12}\varrho^{\mathrm{T}}(t)X\varrho(t) \\ &+ \frac{1}{2}\tau_2^2\dot{x}^{\mathrm{T}}(t)U_1\dot{x}(t) - \int_{-\tau_2}^{0}\int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(s)U_1\dot{x}(s)\mathrm{d}s\mathrm{d}\beta \\ &+ \tau_s\dot{x}^{\mathrm{T}}(t)U_2\dot{x}(t) - \int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(s)U_2\dot{x}(s)\,\mathrm{d}s\mathrm{d}\beta \\ &+ \sum_{1}^{8}\alpha_i(t) \end{aligned} \quad (17)$$

Note that

$$-2\xi^{\mathrm{T}}(t)L\int_{-\tau_2}^{0}\int_{t+\beta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\beta \le \frac{1}{2}\tau_2^2\xi^{\mathrm{T}}(t)LU_1^{-1}L^{\mathrm{T}}\xi(t) + \int_{-\tau_2}^{0}\int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(s)U_1\dot{x}(s)\mathrm{d}s\mathrm{d}\beta \quad (18)$$

$$-2\xi^{\mathrm{T}}(t)H\int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\beta \le \tau_s\xi^{\mathrm{T}}(t)HU_2^{-1}H^{\mathrm{T}}\xi(t) + \int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(s)U_2\dot{x}(s)\mathrm{d}s\mathrm{d}\beta \quad (19)$$

From (17)–(19), it can be seen that

$$\begin{aligned}\dot{V}(x_t) &\le \xi^{\mathrm{T}}(t)\left[\check{\Sigma} + \frac{1}{2}\tau_2^2LU_1^{-1}L^{\mathrm{T}} + \tau_sHU_2^{-1}H^{\mathrm{T}}\right]\xi(t) \\ &- \int_{t-d(t)}^{t} \xi^{\mathrm{T}}(t,s)\Lambda_1\xi(t,s)\mathrm{d}s \\ &- \int_{t-d(t)}^{t-\tau_1} \xi^{\mathrm{T}}(t,s)\Lambda_2\xi(t,s)\mathrm{d}s \\ &- \int_{t-\tau_2}^{t-d(t)} \xi^{\mathrm{T}}(t,s)\Lambda_3\xi(t,s)\mathrm{d}s \end{aligned} \quad (20)$$

where

$$\check{\Sigma} = \begin{bmatrix} \check{\Sigma}_1 & \Sigma_2 \\ \star & \check{\Sigma}_3 \end{bmatrix} + \Upsilon + \Upsilon^{\mathrm{T}} - MA_c - A_c^{\mathrm{T}}M^{\mathrm{T}} + \tau_2 V + \tau_{12}F$$

$$\check{\Sigma}_1 = \mathrm{diag}\{\Pi_{11}, -R_{11}(1-\dot{d}(t)), -W_{11}, -Q_{11}\} + \Psi_1 + \Psi_1^{\mathrm{T}}$$

$$\check{\Sigma}_3 = \mathrm{diag}\left\{\Pi_{22} + \frac{1}{2}\tau_2^2U_1 + \tau_sU_2, -R_{22}/(1-\dot{d}(t)), -W_{22}, -Q_{22}\right\}$$

$$\xi^{\mathrm{T}}(t,s) = [\xi^{\mathrm{T}}(t) \quad \rho^{\mathrm{T}}(s)]$$

Note that $\nu \le \dot{d}(t) \le \mu$, it is easy to obtain that $\check{\Sigma} < \Sigma$. So, if $\Sigma + (1/2)\tau_2^2LU_1^{-1}L^{\mathrm{T}} + \tau_sHU_2^{-1}H^{\mathrm{T}} < 0$ which is equivalent to (4) by Schur complements, and $\Lambda_i \ge 0$ $(i = 1, 2, 3)$, then system (1) is asymptotically stable according to Lyapunov stability theory [29]. □

Remark 1: A new type of augmented Lyapunov functional is introduced in this paper to derive a delay-range-dependent and rate-range-dependent stability condition for linear systems with time-varying delays. Furthermore, the introduction of triple integral terms makes the term $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ in $\zeta(t)$ play an important role in the reduction of conservatism. Whereas in existing literature this term does not contribute to the reduction of the conservativeness such as in [24] or is not included in the augmented vector of the Lyapunov functional such as in [26]. More specifically, if $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ is not introduced in $\zeta(t)$, the introduction of the triple-integral terms does not contribute to further reduction in conservativeness. On the other hand, if only the integral term $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ is introduced in the augmented vector with the triple-integral term omitted in the Lyapunov functional, then this Lyapunov functional also does not lead to a less conservative result.

Remark 2: It is not easy to extend the augmented Lyapunov functional to systems with time-varying delays [26]. One of the key problems is how to estimate the upper bound on the derivative of the Lyapunov functional. In [26], a bounding technique is used to estimate the derivative of $V_{3p}(xt)$ (see (22) in [26]) which introduces some conservativeness and makes the resulting condition unapplicable for $\mu = 0$. However, the method proposed in this paper is much different. In the definition of $\xi(t)$, it is $\dot{x}(t-d(t))(1-\dot{d}(t))$ but not $\dot{x}(t-d(t))$ that is introduced which can absorb some $(1-\dot{d}(t))$. So, it can be seen that only two terms contain $(1-\dot{d}(t))$ in $\check{\Sigma}$, which makes the estimation of the upper bound of the derivative of the Lyapunov functional much easier.

Remark 3: The concerned time-varying delay is assumed to belong to an interval. Furthermore, the delay derivative is also assumed to belong to an interval. The obtained criterion is dependent on both the delay-range and the delay-rate range. The delay derivative in the previous results are usually assumed to be $\dot{d}(t) \le \mu$. Because the lower bound on the delay derivative is not considered, these results may be conservative especially when the lower bound on the delay derivative can be obtained or estimated. It should be noted that the delay-rate range has been considered in [26]. However, the delay-derivative in [26] is restricted to $|\dot{d}(t)| \le \mu$ and $0 < \mu < 1$. So if $-2 \le \dot{d}(t) < 1$, Theorem 3 in [26] is not applicable. Although Corollary 3 in [26] is applicable for this case, it is just a rate-independent criterion which is usually more conservative than rate-dependent ones. Moreover, the restriction on the delay derivative in [26] may result in additional conservativeness when the absolute value of the lower bound on the delay derivative does not equal the upper bound on the delay derivative even if they are both smaller than 1. For example, if $-0.1 \le \dot{d}(t) \le 0.5$ or $-0.2 \le \dot{d}(t) \le 0.5$, Theorem 3 in [26] will deal with these two cases both as $|\dot{d}(t)| \le 0.5$. Clearly, it enlarges the variation range of the delay derivative. So results in [26] may lead to some conservativeness. However, Theorem 1 proposed in this paper can overcome this limitation because it is dependent on the rate-range of the delay.

The information on the lower bound on the delay derivative, ν, has been used in Theorem 1. If this information cannot be obtained or estimated, the assumption on the delay derivative (3) should be changed as $\dot{d}(t) \le \mu < 1$ which is similar to existing assumptions in the literature on the delay derivative. For this case, choose a Lyapunov functional candidate similar to (8) but with $\zeta(t) = \mathrm{col}\{x(t), x(t-\tau_2), \int_{t-\tau_2}^{t} x(s)\mathrm{d}s, x(t-\tau_1)\}$, and define $\xi(t) = \mathrm{col}\{x(t), x(t-d(t)), x(t-\tau_1), x(t-\tau_2), \dot{x}(t), \dot{x}(t-d(t)), \dot{x}(t-\tau_1), \dot{x}(t-\tau_2)\}$, then the following corollary can be obtained following the similar approach to Theorem 1.

Corollary 1: Given scalars μ and $0 \le \tau_1 < \tau_2$, system (1) with a time-varying delay satisfying (2) and $\dot{d}(t) \le \mu < 1$ is asymptotically stable if there exist matrices $U_1 \ge 0$, $U_2 \ge 0$, $P = [P_{ij}]_{4\times4} > 0$, $Q = [Q_{ij}]_{2\times2} \ge 0$, $Z = [Z_{ij}]_{2\times2} \ge 0$, $R = [Z_{ij}]_{2\times2} \ge 0$, $W = [W_{ij}]_{2\times2} \ge 0$, $X = [X_{ij}]_{2\times2} \ge 0$, $\hat{V} = [\hat{V}_{ij}]_{8\times8} \ge 0$, $\hat{F} = [\hat{F}_{ij}]_{8\times8} \ge 0$, $\hat{N}$, $\hat{Y}$, $\hat{S}$, $\hat{H}$, $\hat{L}$, $\hat{M}$ with appropriate dimensions such that the following LMIs hold

$$\begin{bmatrix} \Omega & \frac{1}{2}\tau_2^2\hat{L} & \tau_s\hat{H} \\ \star & -\frac{1}{2}\tau_2^2U_1 & 0 \\ \star & \star & -\tau_sU_2 \end{bmatrix} < 0 \tag{21}$$

$$\hat{\Lambda}_1 = \begin{bmatrix} \hat{V} & \hat{\Gamma}_1 \\ \star & Z \end{bmatrix} \ge 0 \tag{22}$$

$$\hat{\Lambda}_2 = \begin{bmatrix} \hat{F} & \hat{\Gamma}_2 \\ \star & X \end{bmatrix} \ge 0 \tag{23}$$

$$\hat{\Lambda}_3 = \begin{bmatrix} \hat{V}+\hat{F} & \hat{\Gamma}_3 \\ \star & Z+X \end{bmatrix} \ge 0 \tag{24}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \star & \Omega_3 \end{bmatrix} + \hat{\Upsilon} + \hat{\Upsilon}^{\mathrm{T}} - \hat{M}\hat{A}_c - \hat{A}_c^{\mathrm{T}}\hat{M}^{\mathrm{T}} + \tau_2\hat{V} + \tau_{12}\hat{F}$$

$$\Omega_1 = \mathrm{diag}\{\hat{\Pi}_{11}, -R_{11}(1-\mu), -W_{11}, -Q_{11}\} + \hat{\Psi}_1 + \hat{\Psi}_1^{\mathrm{T}}$$

$$\Omega_2 = \begin{bmatrix} P_{11}+\hat{\Pi}_{12} & 0 & P_{14} & P_{12} \\ 0 & P_{44}-(1-\mu)R_{12} & 0 & 0 \\ P_{14}^{\mathrm{T}} & 0 & P_{44}-W_{12} & P_{24}^{\mathrm{T}} \\ P_{12}^{\mathrm{T}} & 0 & P_{24} & P_{22}-Q_{12} \end{bmatrix}$$

$$\Omega_3 = \mathrm{diag}\Big\{\hat{\Pi}_{22} + \frac{1}{2}\tau_2^2U_1 + \tau_sU_2, \\ -(1-\mu)R_{22}, -W_{22}, -Q_{22}\Big\}$$

$$\hat{\Upsilon} = [\hat{N}+\tau_2\hat{L}+\tau_{12}\hat{H} \;\; \hat{Y}-\hat{N}-\hat{S} \;\; \hat{S} \;\; -\hat{Y} \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$\hat{\Pi}_{ij} = Q_{ij} + R_{ij} + W_{ij} + \tau_2Z_{ij} + \tau_{12}X_{ij}, \quad j = 1, 2, i \le j$$

$$\hat{A}_c = [A \;\; A_1 \;\; 0 \;\; 0 \;\; -I \;\; 0 \;\; 0 \;\; 0]$$

$$\hat{\Gamma}_1 = [-\hat{\Psi}_2 + \hat{L} \;\; \hat{N}]$$

$$\hat{\Gamma}_2 = [-\hat{H} \;\; \hat{S}]$$

$$\hat{\Gamma}_3 = [-\hat{\Psi}_2 + \hat{L} + \hat{H} \;\; \hat{Y}]$$

$$\hat{\Psi}_1 = [P_{13}^{\mathrm{T}} \;\; 0 \;\; P_{34} \;\; P_{23}^{\mathrm{T}}]^{\mathrm{T}} [I \;\; 0 \;\; 0 \;\; -I]$$

$$\hat{\Psi}_2 = [P_{33} \;\; 0 \;\; 0 \;\; -P_{33} \;\; P_{13}^{\mathrm{T}} \;\; 0 \;\; P_{34} \;\; P_{23}^{\mathrm{T}}]^{\mathrm{T}}$$

In many circumstances, the information of the delay derivative may not be available. For this case, choose a Lyapunov functional candidate which is similar to (8) but with $R = 0$ and $\zeta(t) = \mathrm{col}\{x(t), x(t-\tau_2), \int_{t-\tau_2}^{t} x(s)\mathrm{d}s, x(t-\tau_1)\}$, and a rate-independent criterion can be obtained from Theorem 1.

Corollary 2: Given scalars $0 \le \tau_1 < \tau_2$, system (1) with a time-varying delay satisfying (2) is asymptotically stable if there exist matrices $U_1 \ge 0$, $U_2 \ge 0$, $P = [P_{ij}]_{4\times4} > 0$,

doi: 10.1049/iet-cta.2008.0508

$Q = [Q_{ij}]_{2\times 2} \geq 0$, $Z = [Z_{ij}]_{2\times 2} \geq 0$, $W = [W_{ij}]_{2\times 2} \geq 0$, $X = [X_{ij}]_{2\times 2} \geq 0$, $\tilde{V} = [\tilde{V}_{ij}]_{7\times 7} \geq 0$, $\tilde{F} = [\tilde{F}_{ij}]_{7\times 7} \geq 0$, $\tilde{N}$, $\tilde{Y}$, $\tilde{S}$, $\tilde{H}$, $\tilde{L}$, $\tilde{M}$ with appropriate dimensions such that the following LMIs hold

$$\begin{bmatrix} \Theta & \frac{1}{2}\tau_2^2\tilde{L} & \tau_s\tilde{H} \\ \star & -\frac{1}{2}\tau_2^2 U_1 & 0 \\ \star & \star & -\tau_s U_2 \end{bmatrix} < 0 \tag{25}$$

$$\tilde{\Lambda}_1 = \begin{bmatrix} \tilde{V} & \tilde{\Gamma}_1 \\ \star & Z \end{bmatrix} \geq 0 \tag{26}$$

$$\tilde{\Lambda}_1 = \begin{bmatrix} \tilde{F} & \tilde{\Gamma}_2 \\ \star & X \end{bmatrix} \geq 0 \tag{27}$$

$$\tilde{\Lambda}_3 = \begin{bmatrix} \tilde{V}+\tilde{F} & \tilde{\Gamma}_3 \\ \star & Z+X \end{bmatrix} \geq 0 \tag{28}$$

where

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \star & \Theta_3 \end{bmatrix} + \tilde{\Upsilon} + \tilde{\Upsilon}^{\mathrm{T}} - \tilde{M}\tilde{A}_c - \tilde{A}_c^{\mathrm{T}}\tilde{M}^{\mathrm{T}} + \tau_2\tilde{V} + \tau_{12}\tilde{F}$$

$$\Theta_1 = \mathrm{diag}\{\tilde{\Pi}_{11}, 0, -W_{11}, -Q_{11}\} + \tilde{\Psi}_1 + \tilde{\Psi}_1^{\mathrm{T}}$$

$$\Theta_2 = \begin{bmatrix} P_{11}+\tilde{\Pi}_{12} & P_{14} & P_{12} \\ 0 & 0 & 0 \\ P_{14}^{\mathrm{T}} & P_{44}-W_{12} & P_{24}^{\mathrm{T}} \\ P_{12}^{\mathrm{T}} & P_{24} & P_{22}-Q_{12} \end{bmatrix}$$

$$\Theta_3 = \mathrm{diag}\{\tilde{\Pi}_{22} + 1/2\tau_2^2 U_1 + \tau_s U_2, -W_{22}, -Q_{22}\}$$

$$\tilde{\Upsilon} = [\tilde{N} + \tau_2\tilde{L} + \tau_{12}\tilde{H} \quad \tilde{Y} - \tilde{N} - \tilde{S} \quad \tilde{S} \quad -\tilde{Y} \quad 0 \quad 0 \quad 0]$$

$$\tilde{\Pi}_{ij} = Q_{ij} + W_{ij} + \tau_2 Z_{ij} + \tau_{12}X_{ij}, \quad j = 1, 2, \ i \leq j$$

$$\hat{A}_c = [A \quad A_1 \quad 0 \quad 0 \quad -I \quad 0 \quad 0]$$

$$\hat{\Gamma}_1 = [-\hat{\Psi}_2 + \hat{L} \quad \hat{N}]$$

$$\hat{\Gamma}_2 = [\hat{H} \quad \hat{S}]$$

$$\hat{\Gamma}_3 = [-\hat{\Psi}_2 + \hat{L} + \hat{H} \quad \hat{Y}]$$

$$\hat{\Psi}_1 = [P_{13}^{\mathrm{T}} \quad 0 \quad P_{34} \quad P_{23}^{\mathrm{T}}]^{\mathrm{T}}[I \quad 0 \quad 0 \quad -I]$$

$$\hat{\Psi}_2 = [P_{33} \quad 0 \quad 0 \quad -P_{33} \quad P_{13}^{\mathrm{T}} \quad 0 \quad P_{34} \quad P_{23}^{\mathrm{T}}]^{\mathrm{T}}$$

3 Numerical examples

Example 1: Consider the following system with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Assume $|\dot{d}(t)| \leq \mu < 1$. For various μ, the maximum upper bounds on delay (MUBDs), τ_2, such that the system is asymptotically stable for a given lower bound, τ_1, are listed in Table 1. Table 1 also compared the results with those in [18, 25, 26] for unknown μ. If the delay derivative does not belong to a symmetric interval, the MUBDs are listed in Table 2 for $\tau_1 = 0$ and various delay derivative intervals. In Table 2, $[a, b]$ means that $a \leq \dot{d}(t) \leq b$. From Table 1 and 2, it can be readily seen that our method achieves much bigger MUBDs.

In Table 2, we also list results obtained by using some special cases of Theorem 1. If $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ is not introduced in the augmented vector $\zeta(t)$ but only the triple-integral terms are introduced in the Lyapunov functional, a special case of Theorem 1 is obtained and is referred to as Corollary 3. Similarly, if the triple-integral terms are not introduced in the Lyapunov functional but only $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ is introduced in the augmented vector, another result can be obtained from Theorem 1 and is referred to as Corollary 4. If the $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ and the triple-integral terms are all removed from the Lyapunov functional, another result can be obtained and is referred to as Corollary 5. Owing to the limitation of space, these three corollaries are omitted in this paper. Applying these three corollaries to this example for asymmetric delay-derivative intervals yields the same results but they are more conservative than those obtained using Theorem 1. This fact illustrates the statement in Remark 1, that is, the $\int_{t-\tau_2}^{t} x(s)\mathrm{d}s$ and the triple-integral terms should co-exist in the Lyapunov functional. This means that both are required to achieve a further reduction of the conservativeness. Furthermore, we can see that our results are still less conservative than those in [26]. This is mainly because a new method is used to estimate the upper bound on the derivative of the Lyapunov functional just as stated in Remark 2 and the information on the lower bound of the delay derivative is fully used in our results just as stated in Remark 3. Specifically, for the four cases of asymmetric delay derivative interval listed in Table 2, He *et al.* [26] deal with them as $|\dot{d}(t)| < 1$, $|\dot{d}(t)| < 0.2$, $|\dot{d}(t)| < 0.1$ and $|\dot{d}(t)| < 0.1$, respectively. Clearly, it enlarges the variation range of the delay derivative. However, both the lower bound and the upper bound on the delay derivative are used in our results. From Table 2, it can be seen that for a given upper bound on the delay derivative, μ, the upper bound on delay, τ_2, increases with the increase of the lower bound on the delay derivative, ν.

Example 2: Consider the following controlled plant

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t)$$

The above plant is assumed to be controlled over a network. The state feedback controller is designed as $u(t) = [-3.75 \ \ -11.5]x(t)$ without considering the effects of the network. When considering the effects of the network, following the method in [9] the closed-loop system is obtained as

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} [-3.75 \ \ -11.5] x(t - d(t))$$

Table 1 MUBDs with given τ_1 for different μ

τ_1	Methods	$\mu = 0.1$	$\mu = 0.5$	$\mu = 0.9$	Unknown μ
0	[25]	3.6053	2.0439	1.3789	1.3454
	[26]	3.6053	2.0439	1.3789	1.3454
	[18]	3.6053	2.0439	1.3789	1.3454
	our results	3.9184	2.7156	2.3877	1.8680
3	[25]	3.6119	3.2234	3.2234	3.2234
	[18]	3.6120	3.2260	3.2260	3.2260
	our results	3.9184	3.3413	3.3413	3.3413
4	[25]	4.0643	4.0643	4.0643	4.0643
	[18]	4.0649	4.0649	4.0649	4.0649
	our results	4.1779	4.1779	4.1779	4.1779
5	[25]	–	–	–	–
	[18]	–	–	–	–
	our results	5.0383	5.0383	5.0383	5.0383
5.300499	[25]	–	–	–	–
	[18]	–	–	–	–
	our results	5.3005	5.3005	5.3005	5.3005

Table 2 MUBDs for asymmetric delay-derivative interval

Methods	[−1, 0.1]	[−0.2, 0.1]	[−0.09, 0.1]	[−0.05, 0.1]
[26]	1.3454	3.0391	3.6053	3.6053
corollaries 3–5	3.6582	3.6582	3.7670	4.0146
theorem 1	3.7660	3.7672	3.9785	4.2536

where $d(t)$ denotes the network-induced delay from the sensor to the actuator and is assumed to satisfy $\tau_1 \leq d(t) \leq \tau_2$. The objective is to determine the maximum allowable transfer interval (MATI), also called the maximum allowable delay bound (MADB), that guarantees the asymptotic stability of the above system. For given $\tau_1 = 0$, MATIs that guarantee the asymptotic stability of the above plant controlled over a network are listed in Table 3. It is clear that the result in this paper is an improvement over those in [9, 26, 30, 31].

Table 3 MATIs obtained by different methods

Methods	MATI
[31]	0.8695
[9]	0.8871
[30]	0.9412
[26]	1.0081
our results	1.0629

4 Conclusion

In this paper, the stability of time-delay systems has been investigated. New delay-dependent stability criteria have been derived by introducing a new type of Lyapunov functional and using the free-weighting-matrices method. The obtained criteria have been shown to be less conservative than existing ones. Numerical examples have been given to illustrate the effectiveness of the proposed method.

5 Acknowledgments

This work is supported by the Beijing Education Committee Cooperation Building Foundation Project XK100070532, China Postdoctoral Science Foundation 20080440308, and National Science Foundation for Distinguished Young Scholars under the Grants 60925011.

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